The Term Structure of Interest Rates in a Heterogeneous Monetary Union *

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Abstract

We build a model to analyze the term structure of sovereign interest rates in a heterogeneous monetary union. We consider two countries, Core and Periphery, in which Core issues default-free sovereign bonds, while Periphery issues defaultable bonds. The possibility of rollover crises in the peripheral bond market implies a default probability that depends on bond supply net of central bank holdings. We decompose yields into term premium and credit risk components. Calibrating the model to Germany and Italy, we find that the model replicates well the asymmetric reaction of the two countries’ yield curves to the ECB’s pandemic emergency purchase program (PEPP) announcement. The flexibility in PEPP purchases over time and across jurisdictions is found to have made a material contribution to the program’s impact on the aggregate area-wide yield curve.

Keywords: sovereign default, quantitative easing, affine model, term structure model.

JEL classification: E5, G12, F45.

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1 Introduction

The sovereign yield curve is a crucial indicator of financing conditions for any given country. For this reason, monetary policy authorities pay great attention to the impact of their actions on the yield curve(s) under their jurisdiction. The importance of yield curves – also known as the “term structure of interest rates” – for the analysis of monetary policy has only increased since the Great Financial Crisis of 2008-09: as (short-term) policy interest rates across advanced economies approached or hit their effective lower bounds, central banks resorted to unconventional tools, such as large-scale asset purchases and forward guidance on the future path of short rates, in order to flatten the yield curve as a way of providing policy stimulus.

Reflecting these developments, term structure models have become important analytical tools, both for central bankers and for scholars interested in monetary policy analysis. In their standard formulation, these models abstract from sovereign default risk, and focus on term risk as the key determinant of yield changes over time and across maturities. While this abstraction is reasonable for jurisdictions in which the perceived risk of sovereign default is small, it is more problematic when focusing on jurisdictions where this risk is a relevant determinant of sovereign yields. A prominent example is the euro area, where sovereign issuers that are viewed as safe coexist – and share a common monetary policy – with other issuers whose bonds display relatively high and volatile credit risk premia.

In this paper, we propose a micro-founded model of the term structure of sovereign interest rates in a heterogeneous monetary union. We build on the influential Vayanos and Vila (2020) term-structure model by extending it in two key dimensions. First, we consider a two-country monetary union setup. Second, we allow for sovereign default risk, which we derive from the optimal choices of a government that faces rollover risk.

Concretely, the monetary union in our model consists of two member states: Core, which issues default-free bonds, and Periphery, which is subject to default risk. As in Vayanos and Vila (2020), our model is populated both by arbitrageurs and preferred-habitat investors. Arbitrageurs trade bonds across both countries and all maturities, while preferred-habitat investors demand bonds with a specific maturity and of a specific jurisdiction. Bond yields in the model are driven by just one stochastic factor, namely the short-term riskless rate.\(^1\) Yields are also affected by the net supply of bonds of each

\(^1\)Although this is a one-factor model, it can be trivially extended to a multifactor environment.
maturity and jurisdiction, by which we mean total bond supply minus the bonds held by the common monetary authority. We treat the supply of bonds by the respective fiscal authorities, and the absorption of bonds by the central bank, as deterministic sequences; this may be interpreted as a situation in which the public sector commits to a particular time path for the net supply of bonds in the market.

We model default by assuming that the market for peripheral bonds is subject to rollover crises in the spirit of Calvo (1988), Cole and Kehoe (2000) and Corsetti and Dedola (2016). When a rollover crisis arrives, the peripheral fiscal authority optimally decides whether to continue servicing its debts or else to partially default on its debt by applying a haircut to bonds of all maturities. We show that, under certain conditions, the peripheral government’s default probability at any given time depends on the present discounted value of the future expected path of fiscal deficits plus redemptions of bonds held by the public, i.e. those not held by the central bank. By purchasing risky sovereign bonds, the central bank decreases the fiscal pressure that the peripheral government would face in the event of a self-fulfilling debt crisis. This is because redemptions of bonds held by the central bank (or interest payments on those bonds) represent payments from the treasury to the central bank which will be rebated back to the treasury through central bank dividends. From the point of view of the consolidated budget constraint of each country, only repayment obligations on bonds held by the public represent fiscal pressure that generates an incentive to default. Therefore, sovereign bond purchases by the central bank reduce the government’s incentives to default when a rollover crisis arrives.\(^2\)

This setup makes the default probability a deterministic sequence, which allows us to derive an affine solution for bond prices, and to decompose bond yields into components related to term risk and – for peripheral bonds – default risk, in the same way as Duffie and Singleton (1999). In particular, we decompose bond yields into four components: (i) an “expectations” term that represents the expected future path of the risk-free rate; (ii) a term premium representing the risk-averse cost of bond price fluctuations; (iii) an

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The notes and codes in this case are available upon request.

\(^2\)We implicitly assume the existence of national central banks –as in the Eurosystem case– and that all asset purchases are actually conducted by each country’s national central bank on behalf of the union-wide central bank. This is broadly consistent with the fact that the large majority of Eurosystem purchases are actually conducted by national central banks, instead of the ECB. Under this assumption, it is indeed the consolidated budget constraint of each country’s fiscal authority and national central bank that matters for the former’s default incentives.
expected default premium to compensate the expected loss due to default; and (iv) a credit risk premium associated with the risk-averse cost of default. In other words, while policy analyses of asset purchase programs (e.g. Li and Wei (2013) or Eser et al. (2019)) have emphasized a “duration extraction” channel, our model distinguishes this from a “default risk extraction” channel that represents the increase in the private sector’s willingness to hold defaultable assets when the quantity of that risk in the market, and the probability of that risk materializing, both decline as a result of central bank asset purchases.

Figure 1: Effects of the pandemic and the PEPP announcement on German, Spanish, and Italian yields

Our framework makes it possible to calculate the impact of unanticipated changes in the time path of net bond issuance. In this paper, we apply our model to analyze the impact of the ECB’s pandemic emergency purchase program (PEPP). The PEPP was announced late on March 18, 2020 with an initial envelope of 750 billion as a response to the tightening in financing conditions across the euro area as a consequence of the Covid-19 crisis. Figure 1 shows that while peripheral yields shifted upwards in February 2020 as the economic impact of the Covid-19 pandemic became clear, this shift was mostly reversed as soon as the PEPP was announced. The surprise nature of
the announcement, in an emergency meeting of the ECB Governing Council, makes it a natural candidate to test our model.

In particular, we calibrate our one-factor model to data on German and Italian yields. We pursue a simple calibration strategy that targets the average shape of both countries' yield curves in the pre-pandemic period, as well as the change in sovereign yield curves in the two days following the PEPP announcement. Despite its parsimony, the model is remarkably successful in reproducing the roughly parallel downward shift in Italian yields, accompanied by a mild flattening of the German yield curve, that followed the PEPP announcement. Our calibration strategy revolves around three key parameters. First, arbitrageurs' risk aversion is identified by matching the German term premium; then the expected loss due to default is identified from the premium on Italian bonds over German bonds; and finally, the impact of net bond issuance on the default probability is identified by explaining the fall in Italian yields when PEPP was announced. Given the relatively high degree of risk aversion needed to explain the German term premium, only a tiny expected loss due to default (roughly four basis points per annum) is needed to explain the observed premium on Italian debt.

In terms of transmission channels, default risk extraction is the most significant channel – much more relevant than the fall in term premia – to explain the behavior of yields in our model. It is particularly relevant for the shape of the response of the Italian yield curve to the PEPP announcement, together with the very asymmetric reaction of the German and Italian curves. The downward shift in the Italian sovereign spread, across all maturities, is explained almost entirely by a lower credit risk premium, driven both by a small decline in the probability of peripheral default and by the reduction in the quantity of defaultable assets that the market must hold. In contrast, the decrease in the expected loss due to default, by itself, plays only a small role in reducing the sovereign spread.

To ensure an adequate response to an asymmetric shock such as Covid, the PEPP was designed to be flexible in the distribution of purchases over time, across asset classes, and across euro area jurisdictions. This flexibility contrasted with the design of the ECB's longer-standing Asset Purchase Programme (APP), which fixed the pace of purchases over time, and allocated purchases across member states according to their "capital keys" – i.e. in proportion to the share of each Eurosystem national central bank in the ECB's capital. We perform counterfactual simulations to evaluate the importance of the PEPP's flexibility for its yield curve effects. We find that the flexible design of
the PEPP program substantially enhanced its impact, and that flexibility in the timing of purchases (frontloading) and flexibility in the allocation across countries (deviations from capital key) complement and reinforce one another. The PEPP announcement reduced Italian yields by around 80bp across the yield curve, with its maximal impact on intermediate maturities. Of this overall effect, we find that roughly 15bp can be attributed to the flexibility of PEPP, as compared with a counterfactual program under which a constant rate of purchases would be allocated across countries according to capital key. Arbitrage implies that the return on riskless bonds is largely determined by the overall level of purchases, regardless of its cross-country distribution. Nonetheless, average euro-area yields depend strongly on how purchases are allocated, because reallocation towards peripheral bonds has a large impact on the yield of peripheral bonds but a negligible impact on that of core bonds.

**Related literature.** This paper provides a link between two different strands of literature. First, we contribute to the finance literature on term structure models. In liquid markets, the action of arbitrageurs tightly links bond returns across maturities and issuers. Ang and Piazzesi (2003), building on Duffie and Kan (1996), constructed an analytical solution for the yield curve in the absence of arbitrage opportunities under the assumption that all yields are affine functions of a set autoregressive Gaussian factors. Vayanos and Vila (2020) showed that an affine term structure model (ATSM) of this type applies to a microfounded setting featuring arbitrageurs with mean-variance utility functions, together with “preferred-habitat” investors whose supply or demand for bonds of specific maturities is linear in those bonds’ yields. This market structure makes it possible to model a wide variety of complex bond market interactions and policy interventions. For example, Greenwood and Vayanos (2014) offered empirical support for the model’s prediction that the price of risk increases as arbitrageurs hold larger maturity-weighted positions; therefore quantitative easing can reduce yields, even if the face value of debt outstanding is unchanged. Further applications include quantitative easing at the effective lower bound (Hamilton and Wu 2012; King (2019)), repo market dynamics (He et al. 2020), and exchange rates (Greenwood et al. 2020; Gourinchas et al. 2020). The bond market structure of Vayanos and Vila (2020) has also been embedded within a New Keynesian model to analyze monetary policy in general equilibrium (Ray 2019). Motivated by the theoretical insights of Vayanos and Vila (2020), a number of papers have incorporated net supply factors into otherwise standard no-arbitrage ATSMs, including Li and Wei (2013) and Eser et al. (2019); the latter paper uses
data on maturity-weighted net issuance of euro-area sovereign bonds (which they call “free-floating duration risk”) to analyze the impact of the APP program.

While they have been widely applied, much of the literature using ATSMs has studied US markets, under the assumption that Treasury securities are nominally riskless. Applications to fixed exchange rate environments – including monetary unions – or to commercial debt make it necessary to consider default risk. Hamilton and Wu (2012) construct an affine term structure model that includes one-period defaultable non-Treasury debt. A key insight in the pricing of defaultable bonds comes from Duffie and Singleton (1999), who show that if the loss caused by default is a fixed fraction of the bond’s value, then the bond-pricing formulas for default-free and defaultable bonds are formally identical, with an adjustment to the discount factor to account for expected losses due to default. Borgy et al. (2012) price defaultable euro-area debt under the assumption that the Duffie and Singleton (1999) condition holds. Altavilla et al. (2015) modeled euro area debt under the assumption that default risk can be priced like any other Gaussian factor. We contribute to this literature in three ways. First, we model explicitly how central bank asset purchases affect the default probability by incorporating the possibility of rollover crises. The mechanism in our paper can be seen as an extension of the two-period economy of Corsetti and Dedola (2016) to a fully dynamic environment. Second, we consider a monetary union, with heterogeneous members, allowing us to analyze spillovers of fiscal and monetary policies in the European context. Third, we show how the non-Gaussian risk associated with default – particularly multi-period, partially-defaultable debt – can be incorporated into an affine term-structure model.

This paper also relates to the literature on monetary-fiscal interactions in the presence of sovereign risk. In contrast to this literature, our attention focuses on the role played by the central bank’s asset purchases in reducing the probability of default, and how they affect the whole term structure of interest rates.

Linking the affine term-structure literature to the literature on sovereign risk is fruitful, because it clarifies that “duration extraction” is neither the only channel, nor the primary channel, by which asset purchases transmit to yields in the European context. Our microfoundations imply that extracting defaultable bonds from private hands re-

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duces the probability of sovereign default. At the same time, as in any preferred-habitat model, reducing the supply of sovereign bonds reduces the yield required to clear that particular market. These two effects reinforce one another in reducing the sovereign premium; our model clearly shows that “default risk extraction” is the predominant impact of asset purchase policy in the Eurozone. The quantitative discipline of the affine term structure framework is crucial here – arbitrage pricing demonstrates that the actual expected loss due to the possibility of default is an order of magnitude smaller than the risk premium demanded by the market to hold defaultable debt when the central bank chooses not to do so.

2 Model

Time is continuous and goes from zero to infinity. We consider a monetary union composed of two countries, Core and Periphery, with a single central bank. The key difference between the two governments is that Core issues risk-free debt whereas Periphery may default on its obligations. We denote Core variables with an asterisk, \( \ast \).

The only assets in the monetary union are a continuum of zero-coupon government bonds, and central bank reserves, which are used as a unit of account (with a constant price of unity). The time-\( t \) price of a bond with maturity \( \tau \) is \( P_t(\tau) \) for peripheral bonds and \( P_t^\ast(\tau) \) for core bonds.

We study a minimalist model of monetary and fiscal interactions that suffices for our focus on the behavior of the yield curve. We assume that the governments and the monetary authorities commit to a fixed time path of bond issuance and bond purchases, as long as no rollover crisis occurs. The one key policy choice that we will endogenize is Periphery’s decision whether to repay or default in case of a rollover crisis.

**Periphery government.** The budget constraint of the peripheral government is

\[
\widehat{d}_t + \widehat{f}_t(0) = \int_0^\infty P_t(\tau) \, \mathcal{I}_t(\tau) \, d\tau + \widehat{\Gamma}_t + \widehat{\Pi}_t,
\]

where \( d_t \) is the primary deficit and \( f_t(0) \) the amount of debt maturing, which must be financed either by issuing new bonds \( \mathcal{I}_t(\tau) \), collecting revenues from seigniorage \( \Gamma_t \) related to central-bank asset purchases, or through emergency taxation \( \Pi_t \). Emergency taxation is zero in normal times, but will be positive during sovereign debt crises, as
described below. The law of motion of the stock of debt is
\[ \frac{\partial f_t(\tau)}{\partial t} = \iota_t(\tau) + \frac{\partial f_t(\tau)}{\partial \tau}, \tag{2} \]
which says that the quantity of bonds of maturity \( \tau \) outstanding at time \( t \), \( f_t(\tau) \), equals the current net issuance of bonds of that maturity, \( \iota_t(\tau) \, dt \), plus the stock of bonds of maturity \( \tau + dt \) that was outstanding at time \( t - dt \).

**Central bank.** Core and Periphery share a common central bank. The central bank’s assets are public bonds and its liabilities are bank reserves. We assume that the central bank maintains separate accounts associated with each national government in the monetary union.\(^4\) The central bank purchases \( \iota^{CB}_t(\tau) \) bonds from Periphery per unit of time. Central bank purchases are financed by issuing interest-paying reserves \( D_t \), which face an infinitely elastic demand from commercial banks. The interest rate on reserves is exogenous and characterized by an Ornstein–Uhlenbeck process,
\[ dr_t = \kappa (\bar{r} - r_t) \, dt + \sigma dB_t, \tag{3} \]
where \( B_t \) is a Brownian motion and \( \kappa \) and \( \bar{r} \) are constants. Similarly, it purchases \( \iota^{CB,*}_t(\tau) \) bonds from Core. As reserves are risk-free, the central bank is in fact buying long-term risky debt in exchange for short-term risk-free debt. The law of motion of the central bank’s portfolio is
\[ \frac{\partial f^{CB}_t(\tau)}{\partial t} = \iota^{CB}_t(\tau) + \frac{\partial f^{CB}_t(\tau)}{\partial \tau}. \tag{4} \]

**Sovereign debt crises.** Similar to Corsetti and Dedola (2016), we focus on self-fulfilling debt crises à la Calvo (1988) or Cole and Kehoe (2000). We assume that investors sometimes, with a certain probability, coordinate on a pessimistic equilibrium in which they stop purchasing peripheral debt. The arrival of this rollover crisis is governed by a Poisson process with rate parameter \( \eta \). If a rollover crisis arrives, the government will be forced to finance its deficits through seigniorage and emergency taxation. At the onset of the crisis, the government decides whether to default on

\(^4\)We assume that the central bank determines seigniorage transfers to each national government in relation to its holdings of that country’s bonds. In practice, this is a roughly realistic depiction of the Eurosystem, in which most bonds are held on national central banks’ own accounts, with only a small fraction of holdings subject to “risk sharing” across national central banks.
its debts or to keep on repaying bonds that mature. Provided that the government decides to repay, the length of the crisis is stochastic, governed by a Poisson process with parameter $\phi$. If instead the government decides to default, it repudiates a fixed fraction $\delta$ of all outstanding bonds, while honoring the remainder.

In order to avoid default during the rollover crisis, the government must honor its flow of expenses without further borrowing. The resulting emergency taxes represent a utility loss for the government. On the other hand, we assume that default imposes a stochastic cost on the government, with c.d.f. $\Phi$. We show in Appendix A.1 that the trade-off between costs of taxation and costs of default implies that the probability of default is $\psi_t = \eta \Phi_t$, where

$$
\Phi_t \equiv \mathbb{P}(\text{default at time } t | \text{crisis}) = \Phi \left( \int_t^\infty e^{-(\hat{r} + \phi)(s-t)} \{ d_s + S_s(0) + \bar{\Gamma} \} ds \right). \quad (5)
$$

Here $\hat{r}$ is the government’s discount factor, $d_s$ represents the deficit, and $S_s(0) \equiv f_s(0) - f_s^{CR}(0)$ represents bond redemptions net of those bonds held by the central bank. $\bar{\Gamma}$ is a constant representing income retained by the central bank to shore up its capital during a rollover crisis. Our assumption is that, during a rollover crisis, the central bank transfers income $\Gamma_s = f_s^{CR}(0) - \bar{\Gamma}$ to the peripheral government, thus returning part of its income from redemptions of maturing bonds back to the national government that issued those bonds.\footnote{As we noted earlier, under current Eurosystem practice, most sovereign bonds are held by the national central banks of the sovereigns that issued them. Much of the income from redemptions of those bonds thus accrues as tax revenue to those sovereigns. Hence our key assumption here is simply that the central bank would not suddenly cut off this revenue flow upon the arrival of a rollover crisis.}

Equation (5) captures the idea that the sovereign default probability is driven by future deficits $d_s$ at times $s \geq t$, but it likewise implies that the central bank can affect the default probability by changing the net future stock of maturing public debt in private hands, $S_s(0)$. Ceteris paribus, if central bank policy decreases the amount of maturing public debt held by the private sector, the sovereign default probability will decrease. This setup allows us to calculate the default probability using only projections.
of future deficits and net redemptions. Under an assumption of perfect foresight about those projections, default is an event that arrives at a known Poisson rate $\psi_t \equiv \eta \Phi_t$. This allows us to apply the framework of Duffie and Singleton (1999) to obtain an affine solution for the term structure, and to decompose bond yields into components related to the dynamics of the risk-free rate and components related to default.

**Core government.** Core, like Periphery, issues bonds and receives seigniorage payments related to purchases of its bonds by the central bank. The stock of core bonds, $f_t^r(\tau)$, and central bank holdings of core bonds, $f_t^{CB*}(\tau)$, follow dynamics formally identical to (2) and (4). The government of Core has a commitment not to default, and is not subject to rollover crises.

**Bond market participants.** We turn now to the pricing of sovereign bonds. We assume there are four classes of players in bond markets. *Preferred-habitat investors* demand bonds of a specific jurisdiction and specific maturity, as an increasing function of the bonds’ yield. Market participants with these characteristics may include pension funds or insurance companies whose liability streams require them to hold assets paying off at specific times in the distant future, or money-market mutual funds that must hold assets that provide liquidity at short horizons. *Arbitrageurs* are willing to hold bonds of any maturity, but their positions are limited by their risk aversion. These players represent liquid, well-informed market participants, such as hedge funds, that have deep pockets but are not willing to take arbitrarily large risks. *Commercial banks* can hold both central bank reserves and short-term bonds. Finally, the consolidated *public sector* (governments and central bank) determines the net supply of bonds.

The *yield* is the spot rate for maturity $\tau$:

$$y_t(\tau) = \frac{-\log P_t(\tau)}{\tau}, \quad y_t^{*}(\tau) = \frac{-\log P_t^{*}(\tau)}{\tau}.$$  

We assume that banks can arbitrage between instantaneous bonds and reserves. Therefore, the short-rate $r_t$ is the limit of the yields $y_t^{*}(\tau)$ of Core bonds when $\tau$ goes to zero. In equilibrium only banks hold reserves, as they are the only agents entitled to do so, while investors and arbitrageurs operate in bond markets.

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6Without these assumptions, the default probability would depend on the future dynamics of bond prices, introducing a nonlinear fixed-point problem that would not, in general, have an affine solution.
We assume that preferred-habitat demand for bonds increases with yield:

\[ Z_t(\tau) = h_t(\tau) - \alpha(\tau) \log P_t(\tau), \quad Z^*_t(\tau) = h^*_t(\tau) - \alpha^*(\tau) \log P^*_t(\tau), \]

where \( \alpha(\tau), \alpha^*(\tau) \geq 0 \) and \( h_t(\tau), h^*_t(\tau) \) are deterministic functions. The net supply of bonds provided by the public sector, \( S_t(\tau), S^*_t(\tau) \) also follows a deterministic function.

Arbitrageurs maximize a mean-variance objective over instantaneous changes in wealth, as in Vayanos and Vila (2020),

\[
\max_{\{X_t(\tau), X^*_t(\tau)\}_{\tau \in (0, \infty)}} \mathbb{E}_t (dW_t) - \frac{\gamma}{2} \text{Var}_t (dW_t)
\]

subject to the law of motion of wealth:

\[
dW_t = \left[ W_t - \int_0^\infty (X_t(\tau) + X^*_t(\tau)) d\tau \right] r_t dt \\
+ \int_0^\infty \left( X_t(\tau) \left( \frac{dP_t(\tau)}{P_t(\tau)} - \delta dN_t \right) + X^*_t(\tau) \frac{dP^*_t(\tau)}{P^*_t(\tau)} \right) d\tau,
\]

where \( \gamma > 0 \) is the representative arbitrageur’s risk-aversion coefficient, and \( X_t(\tau) \) and \( X^*_t(\tau) \) are the nominal quantities of bonds of different maturities held in the arbitrageur’s portfolio. The first term in (6) shows the income from holding bank deposits, while the second term shows the capital gains from holding a portfolio of peripheral bonds \( X_t(\tau) \) and core bonds \( X^*_t(\tau) \), adjusted for the possible arrival of the default event according to a Poisson process \( dN_t \). Note that arbitrageurs can operate in both markets (Core and Periphery), similar to Gourinchas et al. (2020).

Bond market clearing requires consistency between supply and demand for bonds of each maturity and jurisdiction:

\[
S_t(\tau) = Z_t(\tau) + X_t(\tau), \quad S^*_t(\tau) = Z^*_t(\tau) + X^*_t(\tau).
\]

That is, net supply by the public sector equals demand by preferred-habitat investors and arbitrageurs.

**Bond pricing.** We assume that, after default, the peripheral government issues new bonds to replace the defaulted bonds, thus returning to the path of gross bond supply.
to which it is committed. \footnote{Perhaps surprisingly, it would be unrealistic to suppose that debt decreases when default occurs. On the contrary, \citet{Arellano2019} show that debt is more likely to increase following a restructuring. As in their model, default serves here to alleviate short-term fiscal pressure, not to reduce the debt load permanently.} Thus, default leaves the state of the bond market unchanged, so we seek to construct an equilibrium in which bond prices do not depend on previous default events. We conjecture that there exist two pairs of deterministic functions \((A_t(\tau), C_t(\tau))\) and \((A_t^*(\tau), C_t^*(\tau))\) such that the price of bonds can be expressed in log-affine form:

\[ P_t(\tau) = e^{-[A_t(\tau)r_t + C_t(\tau)]}, \quad P_t^*(\tau) = e^{-[A_t^*(\tau)r_t + C_t^*(\tau)]}. \quad (8) \]

Applying Itô’s lemma, the time-\(t\) instantaneous return on an undefaulted bond of maturity \(\tau\) is

\[ \frac{dP_t(\tau)}{P_t(\tau)} = \mu_t(\tau) dt - \sigma A_t(\tau) dB_t, \quad \frac{dP_t^*(\tau)}{P_t^*(\tau)} = \mu_t^*(\tau) dt - \sigma A_t^*(\tau) dB_t, \quad (9) \]

where \footnote{Note that \(\tau\) is a state with dynamics \(d\tau = -dt\), so Itô’s lemma yields derivatives in \(\tau\) as well as \(t\).}

\[ \mu_t(\tau) = \left( \frac{\partial A_t}{\partial \tau} - \frac{\partial A_t}{\partial t} \right) r_t + \left( \frac{\partial C_t}{\partial \tau} - \frac{\partial C_t}{\partial t} \right) - A_t(\tau) \kappa (\bar{r} - r_t) + \frac{1}{2} \sigma^2 [A_t(\tau)]^2, \quad (10) \]

and

\[ \mu_t^*(\tau) = \left( \frac{\partial A_t^*}{\partial \tau} - \frac{\partial A_t^*}{\partial t} \right) r_t + \left( \frac{\partial C_t^*}{\partial \tau} - \frac{\partial C_t^*}{\partial t} \right) - A_t^*(\tau) \kappa (\bar{r} - r_t) + \frac{1}{2} \sigma^2 [A_t^*(\tau)]^2. \quad (11) \]

If we substitute bond returns (9) into the law of motion of wealth (6), we obtain

\[ dW_t = \left[ W_t r_t + \int_0^\infty (X_t(\tau) (\mu_t(\tau) - r_t) + X_t^*(\tau) (\mu_t^*(\tau) - r_t)) d\tau \right] dt, \]

\[ - \int_0^\infty \left[ X_t(\tau) A_t(\tau) + X_t^*(\tau) A_t^*(\tau) \right] d\tau \sigma dB_t, \]

\[ - \int_0^\infty X_t(\tau) d\tau \delta dN_t. \]

Thus, wealth is affected by two very different types of risk: Gaussian variation in
bond prices (seen in the second line of the formula), together with a Poisson risk of losing fraction $\delta$ of the investment in peripheral bonds (third line). The problem of the arbitrageurs takes account of both these risks:

$$\max_{\{X_t(\tau), X^*_t(\tau)\}_{\tau \in (0, \infty)}} \int_0^\infty (X_t(\tau) (\mu_t(\tau) - r_t) + X^*_t(\tau) (\mu^*_t(\tau) - r_t)) \, d\tau$$

$$- \frac{\gamma \sigma^2}{2} \left[ \int_0^\infty (X_t(\tau) A_t(\tau) + X^*_t(\tau) A^*_t(\tau)) \, d\tau \right]^2$$

$$- \psi_t \delta \left[ \int_0^\infty X_t(\tau) \, d\tau \right]$$

$$- \frac{2\psi_t \delta^2}{2} \left[ \int_0^\infty X_t(\tau) \, d\tau \right]^2.$$ 

The first two terms represent the expectation and variance of the component associated with price variation, while the last two terms are associated with default risk, where $\mathbb{E}[\delta dN_t] = \delta \psi_t$ and $\mathbb{V}\text{ar}[\delta dN_t] = \delta^2 \psi_t$.

The first-order conditions are

$$\mu_t(\tau) = r_t + A_t(\tau) \lambda_t + \psi_t \delta + \xi_t,$$  \hspace{1cm} (12)

$$\mu^*_t(\tau) = r_t + A^*_t(\tau) \lambda_t,$$  \hspace{1cm} (13)

where $^9$

$$\lambda_t = \gamma \sigma^2 \left[ \int_0^\infty (X_t(\tau) A_t(\tau) + X^*_t(\tau) A^*_t(\tau)) \, d\tau \right]$$

is the market price of (interest rate) risk and

$$\xi_t = \gamma \psi_t \delta^2 \int_0^\infty X_t(\tau) \, d\tau$$  \hspace{1cm} (14)

is the compensation required by risk-averse arbitrageurs for default risk. Equation (13) shows that the expected growth rate of Core bond prices equals the rate of return on reserves, $r_t$, plus the compensation $A^*_t(\tau) \lambda_t$ for the instantaneous price risk on a bond of a given maturity $\tau$. Analogous terms apply to the expected growth of peripheral bond prices, given by (12), plus the compensation $\psi_t \delta$ for the rate of expected loss due to default, together with the instantaneous default risk premium $\xi_t$.

**Constructing an affine solution.** Market clearing (7) requires that the positions of arbitrageurs equal those of the public sector minus those of the preferred-habitat

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$^9$Our notation in this section follows Vayanos and Vila (2020), except that we have reversed the sign on the variables $\lambda$ and $h$. 

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investors. Therefore, the risk prices $\lambda_t$ and $\xi_t$ must satisfy:

$$\lambda_t = \gamma \sigma^2 \left[ \int_0^\infty \left[ (S_t(\tau) - Z_t(\tau)) A_t(\tau) + (S_t^*(\tau) - Z_t^*(\tau)) A_t^*(\tau) \right] d\tau \right],$$  \hspace{1cm} (15)

$$\xi_t = \gamma \psi_t \delta^2 \int_0^\infty (S_t(\tau) - Z_t(\tau)) d\tau.$$  \hspace{1cm} (16)

Equations (15)-(16) can be used to solve for the unknown coefficients $A_t(\tau)$, $A_t^*(\tau)$, $C_t(\tau)$, and $C_t^*(\tau)$ in the bond price functions. The solution hinges on the observation that if $\psi_t$ is a deterministic function of time, then the left- and right-hand sides of (16) can both be affine functions of $r_t$.\(^{10}\) Indeed, if we consider a perfect foresight scenario for peripheral net bond issuance, then our default framework (5) implies that $\psi_t$ is deterministic. Hence, in this case we can construct an affine solution (8) for prices and yields, in which the risk prices $\lambda_t$ and $\xi_t$ are also affine:

$$\lambda_t = \Lambda_t r_t + \bar{\lambda}_t,$$

$$\xi_t = \Xi_t r_t + \bar{\xi}_t.$$  \hspace{1cm} (17) \hspace{1cm} (18)

Appendix A.2 spells out the affine solution in detail, stating the formulas for the factor loadings $\Lambda_t$ and $\Xi_t$ and intercept terms $\bar{\lambda}_t$ and $\bar{\xi}_t$ consistent with (15)-(16).

3 Monetary policy transmission: analytical results

Our model’s affine solution provides analytical insight into yield curve dynamics and the transmission of conventional and unconventional monetary policy. In this section, we discuss three main findings. First, we decompose the yield curve in order to distinguish the familiar expectations and duration extraction transmission channels of asset purchase policy from our model’s novel sovereign credit risk extraction channel, which arises in a monetary union. Second, we show that conventional monetary policy transmits homogeneously across a monetary union, limiting its scope for stabilizing asymmetric fluctuations. Third, we show that default risk (especially endogenous default risk) is crucial for generating heterogeneous fluctuations in short-term interest rates in a monetary union, like those observed in response to the Covid-19 pandemic shock and the

\(^{10}\)If instead $\psi_t$ is a stochastic process that depends on $r_t$, then there are nonlinear terms on the right-hand side of (16), so the affine solution fails.
Decomposing bond yields. Iterating forward on the first-order condition (12) and using the fact that $P_t(0) = 1$, we can decompose the yield on a peripheral bond of maturity $\tau$ as follows:

$$y_t(\tau) = -\frac{\log P_t(\tau)}{\tau} = \frac{1}{\tau} \int_0^\tau E_t \frac{dP_{t+s}(\tau - s)}{P_{t+s}(\tau - s)} ds = \frac{1}{\tau} \int_0^\tau E_t \mu_{t+s}(\tau - s) ds$$

(19)

$$= \frac{1}{\tau} \mathbb{E}_t \int_0^\tau r_{t+s} ds + \frac{1}{\tau} \mathbb{E}_t \int_0^\tau A_{t+s}(\tau - s) \lambda_{t+s} ds$$

Expected rates $y_t^{EX}(\tau)$

Term premium $y_t^{TP}(\tau)$

$$+ \frac{1}{\tau} \mathbb{E}_t \int_0^\tau \delta \psi_{t+s} ds + \frac{1}{\tau} \mathbb{E}_t \int_0^\tau \xi_{t+s} ds$$

Default premium $y_t^{DP}(\tau)$

Credit risk premium $y_t^{CR}(\tau)$

Therefore, peripheral yields can be decomposed as the sum of four affine components. The default-related components are zero for Core: 11

$$y_t(\tau) = y_t^{EX}(\tau) + y_t^{TP}(\tau) + y_t^{DP}(\tau) + y_t^{CR}(\tau),$$

(22)

$$y_t^{*}(\tau) = y_t^{EX*}(\tau) + y_t^{TP*}(\tau).$$

(23)

The first component, which is equalized across countries, $y_t^{EX}(\tau) = y_t^{EX*}(\tau)$, is the yield in a default-free economy where investors are risk neutral. This is often called the expected rates term, since it is the yield in a default-free economy where the “expectations hypothesis” is true: that is, the bond yield equals the expected value of the short rate over the life of the bond. The second component is the term premium, that is, the compensation required by a risk-averse arbitrageur for holding a bond with a risky price. Since the price process of the defaultable bond is not the same as the price process of the default free bond, the Core and Periphery term premia, $y_t^{TP*}(\tau)$ and $y_t^{TP}(\tau)$, are not exactly equalized. The third component, in the case of peripheral bonds, is the expected default premium, which compensates for the expected loss due

11Equivalently, the bond price can be written as a product of log-affine factors:

$$P_t(\tau) = P_t^{EX}(\tau)P_t^{TP}(\tau)P_t^{DP}(\tau)P_t^{CR}(\tau)$$

(20)

$$P_t^{*}(\tau) = P_t^{EX*}(\tau)P_t^{TP*}(\tau)$$

(21)

where, for each $i \in \{EX, TP, DP, CR\}$, we have $P_t^{i}(\tau) = \exp (-\tau y_t^{i}(\tau))$, and likewise for Core.
to default. Fourth, the yield on peripheral bonds also carries a credit risk premium \( y_t^{CR}(\tau) \), which is the additional return required, beyond the expected default premium, in order for a risk-averse arbitrageur to be willing to hold a defaultable bond. Together, the two components \( y_t^{DP}(\tau) + y_t^{CR}(\tau) \), plus the cross-country difference in term premia \( y_t^{TP}(\tau) - y_t^{TP*}(\tau) \), constitute the yield on a credit default swap.

The decomposition derived here shows how the transmission of monetary policy operates via four different channels. First, it operates through anticipated changes in the future path of interest rates (forward guidance). Second, it operates through “duration extraction”, by which central bank bond purchases reduce the market price of interest rate risk, as in Vayanos and Vila (2020). Third, policy transmits through changes in the expected default premium, as central bank purchases reduce the likelihood of self-fulfilling debt crises as explained above, as first identified by Corsetti and Dedola (2016). Finally, it transmits through changes in “default risk extraction”, by which central bank bond purchases reduce the market price of default risk, both because there is less defaultable debt in the market, and because the probability of default on that debt is lower.

**Conventional monetary policy transmission.** First we discuss conventional monetary policy transmission in a monetary union. For simplicity, we focus on the stochastic steady state of the model, in which the short rate \( r_t \) is stochastic, but there is no further time variation in the model’s parameters. We suppress time subscripts wherever possible when analyzing the stochastic steady state. Following Vayanos and Vila (2020), we can express

\[
A^*(\tau) = \frac{1 - e^{-\hat{\kappa}\tau}}{\hat{\kappa}}, \quad A(\tau) = \frac{1 + \Xi - e^{-\hat{\kappa}\tau}}{\hat{\kappa}},
\]  

(24)

where

\[
\hat{\kappa} = \kappa - \Lambda = \kappa + \gamma \sigma^2 \int_0^\infty \left( \alpha(\tau) \left( \frac{1 + \Xi - e^{-\hat{\kappa}\tau}}{\hat{\kappa}} \right)^2 + \alpha^*(\tau) \left( \frac{1 - e^{-\hat{\kappa}\tau}}{\hat{\kappa}} \right)^2 \right) d\tau,
\]

is the risk neutral counterpart of \( \kappa \), and \( \Xi \) is the steady state value of the loading of the default risk price \( \xi_t \) on the short rate (eq. 18).

This solution shows that conventional monetary policy has roughly uniform effects across the monetary union. In particular, the reaction of the instantaneous forward
rate, \( i_t(\tau) \equiv -\frac{\partial \log(P_t(\tau))}{\partial \tau} \), is identical in Core and Periphery:

\[
\frac{\partial i_t(\tau)}{\partial r_t} = -\frac{\partial}{\partial r_t} \frac{\partial \log(P_t(\tau))}{\partial \tau} = e^{-\hat{\kappa} \tau} = \frac{\partial i_t^*(\tau)}{\partial r_t}.
\]

If we instead consider the yield curve itself, then the initial response to a monetary policy shocks at time \( t \) is:

\[
\frac{\partial y_t^*(\tau)}{\partial r_t} = 1 - \frac{e^{-\hat{\kappa} \tau}}{\tau \hat{\kappa}} > \frac{1 + \Xi - e^{-\hat{\kappa} \tau}}{\tau \hat{\kappa}} = \frac{\partial y_t(\tau)}{\partial r_t},
\]

(this response subsequently decays at rate \( \kappa \) with the time since the shock). Since \( \Xi < 0 \), the reaction of peripheral yields is damped in comparison to that of the core. But in practice, this difference is negligible. If the preferred habitat slope \( \alpha(\tau) \) or the default probability \( \psi \) is sufficiently close to zero, then \( \Xi \approx 0 \), so the responses of the two yield curves are approximately equal. In the quantitative section below we will see that the data reject a highly elastic preferred habitat demand curve (large \( \alpha(\tau) \)) and imply a small value for \( \psi \). Hence our calibrated model implies that the impact of conventional monetary policy on core and peripheral yields is virtually indistinguishable.

**What drives the short end of the yield curve?**

For a country without default risk, the shortest yield coincides with the interest rate on reserves:\(^{12}\)

\[
\lim_{\tau \to 0} y_t^*(\tau) = \lim_{\tau \to 0} A^*(\tau) r_t + C^*(\tau) = \lim_{\tau \to 0} \frac{1 - e^{-\hat{\kappa} \tau}}{\hat{\kappa}} r_t + \int_0^\tau A^*(\tau) \kappa r - \frac{1}{2} \sigma^2 [A^*(\tau)]^2 + \lambda A^*(\tau) d\tau
\]

\[
= \lim_{\tau \to 0} e^{-\hat{\kappa} \tau} r_t + A^*(\tau) \kappa r - \frac{1}{2} \sigma^2 [A^*(\tau)]^2 + \lambda A^*(\tau) = r_t,
\]

where the second line applies L'Hôpital's rule and the fact that \( A^*(0) = 0 \). In this case, as in the original Vayanos and Vila (2020) model, changes in structural parameters can produce changes in the slope of the yield curve, but never parallel shifts, as the short end of the curve is pinned down by the short-term rate.

Hence, if we abstract from default, our model cannot reproduce yield curve shifts like those observed in Europe in the context of Covid-19 and the PEPP announcement (see Figure 1 above). However, if we consider a country with default risk, the yield

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\(^{12}\)Here again, for simplicity, our analysis focuses on the model’s stochastic steady state, but our findings also apply to the general, nonstationary version of the model.
curve is given by

\[
y_t(\tau) = \frac{A(\tau) r_t + C(\tau)}{\tau} = (\psi \delta + \bar{\xi}) + \frac{1 + \frac{1 - e^{-\kappa \tau}}{\kappa} r_t + \int_0^\tau A(\tau) \kappa \bar{\rho} - \frac{1}{2} \sigma^2 [A(\tau)]^2 + \bar{\lambda} A(\tau) d\tau}{\tau}.
\]

Note that the default-related term \(\psi \delta + \bar{\xi}\) is independent of maturity \(\tau\), so this term produces a parallel shift in the yield curve when any of its components change. Hence, changes in the default probability \(\psi\) or the default risk compensation term \(\bar{\xi}\) affect even the shortest yields:

\[
\lim_{\tau \to 0} y_t(\tau) = (\psi \delta + \bar{\xi}) + \lim_{\tau \to 0} e^{-\kappa \tau} r_t + A(\tau) \kappa \bar{\rho} - \frac{1}{2} \sigma^2 [A(\tau)]^2 + \bar{\lambda} A(\tau) = r_t + (\psi \delta + \bar{\xi}).
\]

Thus, default risk generates a spread over the risk-free short rate. The spread includes the expected default premium \(\psi \delta\). The second term is the intercept \(\bar{\xi}\) of the credit risk premium \(\xi\) in equation (18), given by

\[
\bar{\xi} = \gamma \psi \delta^2 \int_0^\infty (S(\tau) - h(\tau) - \alpha(\tau) C(\tau)) d\tau. \tag{25}
\]

Equation (25) shows that changes in the default rate \(\psi\), the expected loss \(\delta\) or the risk aversion parameter will, ceteris paribus, produce a shift in the peripheral yield curve in this monetary union. Asset purchases will also shift the peripheral yield curve, including the shortest yields, through two channels. First, they reduce the default rate through equation (5). Second, they extract default risk from the balance sheets of arbitrageurs (reducing the quantity \(S(\tau)\) that arbitrageurs must hold).

This suggests that our model of endogenous default risk has potential to explain the yield curve dynamics seen over the course of the Covid-19 crisis. We calibrate our model to perform a quantitative evaluation of the PEPP announcement, which will then provide the basis for a counterfactual analysis of alternative policies.
4 Quantitative analysis

4.1 Steady state calibration

We calibrate the model taking the two countries in the union, Core and Periphery, to represent Germany and Italy. The yield curve decomposition in equations (22)-(23) suggests a clear strategy to identify the key parameters of the model, in three simple steps. First, the risk aversion parameter $\gamma$ can be identified from the German term premium. Second, the expected loss from default can be identified from the sovereign premium on Italian debt. Third, the responsiveness of the default probability to the amount of free-floating duration risk can be identified from the impact of the initial PEPP announcement on Italian and German yields. These parameters are identified conditional on an inelastic preferred-habitat demand for bonds, $\alpha^*(\tau) = \alpha(\tau) = 0$. The asset purchase effects we identify are largely unaffected by $\alpha$, over a wide range of possible values of this parameter.

The first two calibration steps only require a look at the longer-term average behavior of yields, leaving for later the impact of PEPP. Therefore we first consider pre-pandemic yields data in light of our model’s steady-state term structure; we take “steady state” to mean the effective lower bound period 2013-2019, and we use the subscript “ss” to denote average behavior over this period.

We parameterize the model with a single stochastic factor: the riskless short yield $r_t$. We take daily data from Datastream on one-month, one-year, five-year and ten-year zero-coupon sovereign bond yields for Germany and Italy. We set the mean of $r_t$, $\bar{r} = -49$bp, and the standard deviation, $\sigma = 32$bp, equal to those of the one-month German yield over 2013-2019. We set $\kappa = 0.12$, equivalent to a monthly autocorrelation of 0.99. We assume that Core and Periphery issue bonds with maturities from one month to 10 years.\(^{13}\)

For our steady-state analysis, we assume that the net supplies of German and Italian bonds, $f^*_{ss}(\tau)$ and $f_{ss}(\tau)$, are fixed at their average nominal values (in billions of euros) from the 2013-2019 period, taking data on gross government debt from Eurostat’s Government Finance Statistics and data on gross asset purchases from the ECB’s online documentation of its PSPP program. We assume that the fraction of net debt in the

\(^{13}\)We fix $\rho$ instead of taking it from the data because over long samples, the German one-month yield suffers large regime shifts, making the empirical value of $\rho$ depend on the sample chosen. The true process for German short yields appears more complex than the simple AR(1) we impose here.
hands of arbitrageurs (rather than preferred-habitat investors), \( \frac{X_{ss}(\tau) + X^*_{ss}(\tau)}{S_{ss}(\tau)} \), is 56%, which is consistent with the holdings of gross debt reported by Eser et al. (2019), Table 1, for 2013-2019. Together with an assumption that bond supply is uniform across maturities over this period, these data nail down the intercept terms \( h_{ss}(\tau) \) and \( h^*_{ss}(\tau) \) in the preferred-habitat demand equations.

Now note that in our model, the default premia \( y^{DP*}(\tau) \) and \( y^{CR*}(\tau) \) are zero at all maturities (eq. 23) \( \tau \) for Core (i.e., Germany). The term premium \( y^{TP*}(\tau) \) is zero when \( \gamma = 0 \), and increases with \( \gamma \). Hence it is natural to calibrate \( \gamma \) by matching the German term premium. Given the dynamics of the risk-free rate \( r_t \) and net asset holdings described above, we find that setting \( \gamma = 0.16 \) best fits the German 10-year term premium. The fit is illustrated in the left panel of Figure 2, which shows the model-generated steady-state German yield curve (solid line), together with the expectations component \( y^{EX*}_{ss}(\tau) \) (dotted line); hence the difference between the dotted and solid lines is the term premium component \( y^{TP*}_{ss}(\tau) \).\(^{14}\) The stars show mean German yields, 2013-2019, for 1m, 1Y, 5Y and 10Y maturities.

Next, conditional on \( \gamma \), we estimate the expected loss due to default from the premium on Italian bonds over German bonds for the same years. First, we need to calibrate the haircut \( \delta \) in case of default, which we set to 0.25, a reasonable number given the international evidence (see Cruces and Trebesch (2013)). Given \( \delta \), we estimate the steady-state value of the default probability, \( \psi_{ss} = \eta \Phi_{ss} \) by matching the observed sovereign premium (specifically, we target the mean of the Italian 5Y and 10Y premia over the period 2013-2019). The estimated value is \( \psi_{ss} = 14 \)bp, meaning that the expected cost of default, \( \delta \psi_{ss} \), is less than four basis points annually, explaining only a small fraction of the sovereign premium seen in the right panel of Figure 2. In the figure, the expected default cost component \( y^{DP}_{ss}(\tau) \) can be seen as the distance between the dash-dot and dashed lines. The much larger component of the sovereign premium is the credit risk premium \( y^{CR}_{ss}(\tau) \), which is the distance between the dashed and solid lines in the figure. The red stars in the figure show empirical Italian yields.

\(^{14}\)The mean one-month German yield over our calibration period is -49bp. Since Figure 2 depicts the yield curves at their steady state configuration, the expectations component \( y^{EX*}_{ss}(\tau) \) is flat, reflecting a constant expected short rate of -49bp. This component would not be flat if we graphed the yield curves in a different situation, where the stochastic factor was not currently at its mean value.
Figure 2: Decomposing model-generated yield curves: Germany and Italy

Notes.

Left panel. Stars: average yields (annualized, basis points) on zero-coupon 1m, 1Y, 5Y and 10Y German sovereign bonds, 2013-2019. Source: Datastream.

Blue lines: Decomposition of model-generated steady-state German yield curve $y^*(\tau)$ into expectations component (dotted) plus term premium (solid).

Right panel. Stars: average yields (annualized, basis points) on zero-coupon 1m, 1Y, 5Y and 10Y Italian sovereign bonds, 2013-2019. Source: Datastream.

Red lines: Decomposition of model-generated steady-state Italian yield curve $y(\tau)$ into expectations component (dotted), plus term premium (dash-dotted), plus expected default premium (dashed), and plus credit risk premium (solid).
4.2 Dynamics: modelling the impact of PEPP on the yield curve

Given our calibration from the pre-pandemic period 2013-2019, we next consider how net bond supply affects the default probability by examining the impact of the PEPP announcement on peripheral yields. Strikingly, the effect of the PEPP announcement on yields looks almost like a mirror image of the preceding shifts caused by the pandemic, as Figure 1 showed. As the spread of the pandemic became clear to market participants, the Spanish and Italian yield curves shifted up in a roughly parallel fashion, with a modest shift downwards in German yields (the left panel of the figure compares average yields over the week of Feb 13-19 with yields one month later, over the week of 12-18 March, just before the PEPP announcement). Much of this rise was reversed, by roughly parallel downward shifts in Spanish and Italian yields, upon the announcement of PEPP (the right panel compares end-of-day yield curves for March 18 and 20, before and after the announcement).

As the pandemic took hold in Italy, Spain, and the rest of Europe, it became clear that governments would be obliged to undertake a massive fiscal response, implying higher gross debt levels. On the other hand, the PEPP announcement revealed that much of this new debt would be taken onto the Eurosystem’s balance sheet. In any model of risk-averse arbitrage, these changes in net supply would imply a rise in yields in response to news of the pandemic, and a fall in yields upon the announcement of PEPP. But in our model, the direct effect of net supply on yields is reinforced by an additional channel going from net supply to the default probability and finally to yields.

Therefore, it is natural to estimate the effect of net bond issuance on the default probability by matching the fall in Italian yields when PEPP was announced. To do so, let us define time-\( t \) fiscal pressure \( F_t \) as the discounted sum of future deficits and net bond redemptions:

\[
F_t \equiv \int_0^{\infty} e^{-(\hat{r} + \phi)(t+s)} \left( d_{t+s} + f_{t+s}(0) - f_{CB}^{CB}(0) \right) ds.
\]

We set \( \hat{r} = 0 \) and \( \phi = 0.5 \), so that the expected duration of a rollover crisis is two years. We assume that the cdf \( \Phi(\cdot) \) is distributed according to a uniform random variable over an interval \([\mathcal{E}, \mathcal{F}]\) sufficiently wide to include all the fiscal scenarios we consider. Then,
the unconditional default probability can be written as

\[ \psi_t = \psi_{ss} + \theta (F_t - F_{ss}). \]  

where \( \theta \equiv \eta \Phi' \) equals the arrival probability \( \eta \) of a rollover crisis times the density \( \Phi' \) of the default costs distribution \( \Phi \), and the subscript \( ss \) represents the “steady state” period 2013-2019, as before.\(^{15}\)

To estimate \( \theta \), we take long-term Banco de España forecasts of German and Italian gross debt, conditional on June 2020 data. We simulate a no-PEPP scenario, \( \{F_{t+s}^{before}\}_{s \geq 0} \), in which ECB purchases are determined by the earlier APP program, and a scenario \( \{F_{t+s}^{after}\}_{s \geq 0} \) in which the APP program is complemented by the program of PEPP purchases, as announced on March 18, 2020, interpreting the difference as the effect of the announcement. In other words, the PEPP announcement in our simulation represents a revision of the sequence \( f_{t+s}^{CB}(0) \) in (26) from a scenario that included APP only, to a scenario that includes both APP and PEPP, while keeping fixed the sequences \( d_{t+s} \) and \( f_{t+s}(0) \). Thus, our analysis of the PEPP announcement can be understood as a variation of monetary policy around an initial policy path that includes the fiscal policies induced by the pandemic and the expansion of APP announced on March 12, 2020.

We focus on the effects of PEPP as it was originally announced in March, with an overall envelope of 750 billion euros to be spent over the course of 2020, rather than attempting to model subsequent announcements (at the end of June, and in December) of larger purchase envelopes to be spent over a longer horizon.\(^{16}\) Since the path of purchases under PEPP was flexible, rather than pre-defined like the APP, we must make some assumptions about arbitrageurs’ expectations for the PEPP program at the time of announcement. Our baseline scenario assumes that arbitrageurs anticipated PEPP purchases through June 2020 with perfect foresight—implying some frontloading, and some excess purchases of Italian debt, compared with the Italian capital key (the upper left panel of Figure 6 graphs purchases in this scenario). We assume that from July to December, PEPP purchases were expected to accrue at a constant pace, up to the

\(^{15}\)The assumption of uniformity implies that the value of \( \theta \) inferred by matching the impact of the PEPP announcement can also be used to infer the impact of the alternative policies we consider below.

\(^{16}\)Subsequent PEPP announcements (at the end of June, and in December) were, to a large extent, anticipated by the market according to different surveys.
Original PEPP envelope, while maintaining the deviations from capital key that were observed through June. We then estimate the parameter $\theta$ to fit the shift in the Italian yield curve when expectations are revised, following the March 18 announcement, from the no-PEPP scenario to the PEPP scenario.\footnote{We estimate the response of the default probability to fiscal pressure, $\theta$, without attempting to identify the arrival rate of a rollover crisis, $\eta$, separately from the derivative $\Phi'$.}

The results under the estimated parameter value $\theta = 2.02 \times 10^{-5}$ are shown in Figure 3, where stars indicate the change in yields between March 18 (pre-announcement) and March 20, 2020 (after the PEPP announcement).\footnote{We take the change from March 18 to 20 as our measure of the impact of the PEPP announcement.}

\begin{itemize}
\item \textbf{Left panel.} Blue stars: shift in German yields, 18 to 20 March 2020 (Datastream).
\item Blue lines: model decomposition of shift in German yields into expectations component (dotted) plus term premium (solid), in response to purchase announcement shown in top, left panel of Fig. 6.
\item \textbf{Right panel.} Red stars: shift in Italian yields, 18 to 20 March 2020 (Datamap).
\item Red lines: model decomposition of shift in Italian yields into expectations component (dotted), plus term premium (dashed-dotted), plus expected default premium (dashed), and plus credit risk premium (solid), in response to purchase announcement shown in top, left panel of Fig. 6.
\end{itemize}
show that the PEPP announcement had a small, hump-shaped impact on German yields, which rose at a one-year maturity and fell at a ten-year maturity. In contrast, the impact on Italian yields was dramatic (right panel, red stars), showing a hump-shaped decline that had its largest impact, of almost 90 basis points, at a five-year maturity. Hence, across all maturities, the announcement was associated with a large reduction in average eurozone bond yields, together with a sharp drop in cross-country differentials.

Our model does a good job of reproducing the effect on yields in both countries. Given the degree of risk aversion implied by our calibration, the PEPP announcement is predicted to cause a modest decline in the German term premium, of slightly under 10 basis points. The effect on the Italian term premium is similar, but the sovereign premium on Italian debt declines sharply, because the increased absorption of duration risk and credit risk makes the market much more willing to take part of this risk into its own hands. Under our parameter estimate, the PEPP purchases cause a small decline in the expected default costs on Italian debt, on the order of one basis point per annum, seen as the difference between the dashed and dash-dot lines in the right panel of Fig. 3. While this change is small in absolute terms, it represents a nontrivial reduction in the already small level of default risk on Italian debt (4bp per annum at the model’s steady state). Therefore, the direct and indirect effects of purchases jointly cause a large decrease in the credit risk premium $y_t^{CR}(\tau)$, which accounts for the largest share of the response to the PEPP announcement.

The reduction in Italian yields at the time of the announcement is large at all maturities, but is strongest for bonds of intermediate duration, which will be maturing when cumulative net purchases are still large. In contrast, one month bonds mature before many purchases have taken place. At the opposite extreme, for ten-year bonds, our scenario implies that most net redemptions will have occurred, and hence yields will be rising again, by the time the bonds mature. Since yields are forward-looking, the future return to normality limits the change in 10-year yields on impact.

We can also validate our model by studying the effect of the PEPP announcement on CDS spreads, shown by the green stars in Figure 4. When a default (or related credit event) occurs, a CDS pays off the difference between the face value of the defaulted bond and the maturity value of the CDS. Since the model treats the riskless short rate as an exogenous factor, changing the path of purchases has no impact on the expectations component $y_t^{EX}(\tau)$.

19 Because yields were still volatile across Europe on the 19th, but settled down from the 20th onwards.

19 Since the model treats the riskless short rate as an exogenous factor, changing the path of purchases has no impact on the expectations component $y_t^{EX}(\tau)$. 

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Figure 4: Impact of PEPP on sovereign premium and CDS spreads

Notes.
Effects of the PEPP announcement on the sovereign premium and CDS spreads (March 18-20, 2020). Black stars show the change in the sovereign premium on Italian over German debt (1 month, 12 month, 5 year, and 10 year maturities). Green stars show the change in the Italian CDS spread (6 month, 12 month, 5 year, and 10 year maturities). The solid blue line shows the model-generated change in the sovereign premium.

A bond and its remaining market value. Hence the payoff to a CDS is equal to that of a portfolio that is long one default-free bond and short one defaultable bond. Ignoring transactions costs and assuming absence of arbitrage, this implies that the per-period price of holding a CDS (the spread that is paid as a premium on the CDS contract) must equal the price of holding such a portfolio (Duffie (1999)) which in our case is simply the sovereign spread, that is, the yield on an Italian bond minus the yield on a German bond. Hence, abstracting from transactions costs and arbitrage opportunities, upon the announcement of PEPP, CDS spreads and the Italian-German sovereign spread should fall by the same amount. This prediction is close to being true at 5-year and 10-year maturities, but fails at short maturities, where the observed decrease in the CDS spread is substantially smaller than the observed decline in the sovereign premium.
Figure 5: Persistence of PEPP effects: baseline scenario.

Notes.

Left panel. PEPP effects on German yields over time, for one month, one year, five year, and ten year maturities, under the baseline purchase program shown in top, left panel of Fig. 6.

Right panel. PEPP effects on Italian yields over time, for one month, one year, five year, and ten year maturities, under the baseline purchase program shown in top, left panel of Fig. 6.

Interestingly, our model’s prediction for the impact of PEPP lies mostly between the observed impacts on CDS spreads and sovereign spreads. In particular, the relatively small decrease in one-month spreads predicted by our model is more consistent with the CDS data than it is with the data on the sovereign premium.

Beyond its powerful effect upon announcement, our model also implies that PEPP has a persistent effect on yields over time. Figure 5 illustrates the impulse responses of yields to the PEPP announcement, for one-month, 12-month, 60-month, and 120-month maturities, under the assumption that the purchase program unfolds as expected under our baseline scenario. The small decrease in German yields (left panel) mostly affects long bonds, and decays smoothly, with a half-life of roughly three years. The much larger reduction in Italian yields (right panel) is very persistent, but differs across durations. The impact on 10-year bonds declines smoothly over time, while for one-month and one-year bonds the effect is initially increasing, building up to a reduction in yields of almost a full percentage point in the second year after the announcement.
To understand why these responses vary with duration, note that PEPP purchases are designed to be “market neutral”, meaning that bonds are bought in proportions that roughly reflect the supply available in the market. The actual supply available is approximately uniform across maturities, so our simulations are based on the assumption that both gross public issuance and gross ECB purchases are likewise uniform. Also reflecting ECB practice, we assume purchases are held to maturity. The effect on short yields increases over the course of 2020, because the quantity of short bonds held increases over the course of 2020, cumulating new purchases with bonds purchased earlier at slightly greater maturity. The trough in short yields occurs just as gross purchases cease. From this time onwards, the whole portfolio gradually matures; the decrease in 10-year yields from 2021 onwards is only due to arbitrage across durations, not because the program still holds any 10-year bonds. As the average maturity of the PEPP portfolio shortens, its impact on long yields fades away, followed by its impact on short yields. The final effects of the program disappear as the last bonds mature, 120 months after the end of gross purchases.

5 Counterfactual experiments: the role of flexibility in purchases

Since our assumption that the peripheral default probability varies with the quantity of free-floating duration risk appears to match the impact of PEPP well, across jurisdictions and maturities, we now apply our model to evaluating counterfactual scenarios. In particular, we are interested in evaluating how the flexible design of the PEPP purchases altered their impact, relative to the design of earlier ECB programs. Figure 6 illustrates four of the policy comparisons we will consider.

The top, left panel shows the baseline scenario that we used in Section 4.2 as a stand-in for expectations of the PEPP program upon announcement. The black line shows our scenario for cumulative PEPP purchases of German sovereign bonds (expressed in face value, in billions of euros) for months 3-12 (indicating March-December, 2020). Likewise, the red line shows our scenario for Italian purchases. The path of purchases up to the end of June represents actual PEPP purchases, which accumulated almost linearly over time, at a pace that, if continued, would have exhausted the envelope before the end of the year. As a fraction of the monthly total, Italian purchases exceeded Italy’s
Notes.

Top, left panel: Baseline model scenario for PEPP purchase expectations as of March 2020. Blue circles: Germany; red squares: Italy; black: aggregate face value. Effect on yields is shown in Figs. 3-5.

Top, right panel: Comparing baseline PEPP scenario vs. inflexible “APP-style” scenario with a constant pace of purchases and allocations equal to capital keys. Blue circles: Germany; red squares: Italy; black: aggregate face value. Effect on yields is shown in Fig. 7.

Bottom, left panel: Comparing inflexible “APP-style” scenario with a scenario that reallocates purchases by ±5%. Blue circles: Germany; red squares: Italy; black: aggregate face value. Effect on yields shown in top panel of Fig. 8.

Bottom, right panel: Comparing inflexible “APP-style” scenario with a “frontloading” scenario that completes all purchases by July. Blue circles: Germany; red squares: Italy; black: aggregate face value. Effect on yields shown in middle panel of Fig. 8.
capital key, while purchases of German bonds were close to capital key (purchases of French bonds were substantially below capital key). Since our scenario is intended to model the effects of the initial announcement, we abstract from the actual path of purchases after June (when a recalibration was announced), and instead suppose that purchases from July onwards would use up the remaining PEPP envelope at a constant pace, while maintaining the initial deviations in capital key. The green line represents cumulative purchases of the whole PEPP program in our two-country model – that is, it is the sum of the black and red lines. (The blue line on the x-axis represents no PEPP, which is the basis of comparison in Figs. 3-5.)

We compare our PEPP scenario to a hypothetical alternative following the inflexible design principles of the earlier Asset Purchase Programme (APP). That program imposed a constant pace of purchases to exhaust the intended envelope by the fixed end date of the program, and allocated purchases according to each eurozone state’s capital key. This comparison is illustrated in the top, right panel of Fig. 6. There, the black dashed line represents a constant rate of German bond purchases from March to December that add up to Germany’s capital-key share (26.4%) of the PEPP envelope. Likewise, the red dashed line represents purchases of Italian bonds, at a constant rate, that add up to Italy’s capital-key share (17.0%) of the PEPP envelope. The blue dashed line represents total cumulative purchases under our “APP-style” counterfactual, so it is the sum of the black and red dashed lines. As before, the black, red, and green solid lines represent cumulative purchases under our PEPP scenario (German, Italian, and total bonds, respectively); the arrows indicate how this scenario deviates from APP principles. Clearly, our PEPP scenario imposes frontloading, with an initial pace of purchases faster than the APP design would permit. Simultaneously, our PEPP scenario allocates more purchases to Italy than the APP design would, while total purchases of German debt in our PEPP scenario are similar to those in our APP scenario (close to capital key). Hence total PEPP purchases (green solid line) end up above the intended envelope (blue dashed line) since our two-country simulation abstracts from the jurisdictions where purchases were lowest, relative to capital key.

Figure 7 shows how the effects of the purchase program differ between the PEPP and APP designs. The figure shows the yields resulting from the PEPP scenario minus those under the APP scenario, so the fact that the differences shown in the graphs

\[20\]While the total PEPP envelope was 750 billion euros, we only analyze the part that was dedicated to sovereign bonds (608 billion), abstracting from private-sector and supranational purchases.
Figure 7: Comparing impact of PEPP scenario with “inflexible” alternative

Notes.
Panels show the difference between the baseline PEPP scenario and an “APP-style” scenario that imposes a constant pace of purchases and allocations equal to capital keys, as illustrated in top, right panel of Fig. 6.

are negative indicates that the PEPP design reduces yields more than the APP design does. The left column (like Fig. 3) decomposes the effects on the yield curve at the time of the announcement; the top row refers to Germany, while the lower row refers to Italy. The PEPP design causes a small extra reduction in German yields, by half a basis point at longer maturities, compared to APP. But the reduction in yields is much more significant in the Italian case, where PEPP shifts the yield curve by almost fifteen additional basis points at most maturities, compared with the APP design. Most of the difference between PEPP and APP is attributable to a decline in the credit risk premium (the distance between the dashed and solid red lines in the lower, left panel of the figure). In the right column (as in Fig. 5), we see that the additional impact of the flexible PEPP design is persistent. Flexibility decreases ten-year yields by more than
five basis points over four years, while at the short end, yields decrease by more than five basis points for roughly eight years.

Which aspect of the flexibility of PEPP is most important for its stronger impact on yields, as compared with an APP design? We next perform additional counterfactual experiments to distinguish the role of flexibility in the allocation of purchases from flexibility in their timing. The first panel of Figure 8 documents the effect of allocation alone. Taking the APP design as a starting point, it considers the impact of reallocating purchases worth five percentage points of the overall PEPP envelope from Core to Periphery. This change in purchases is illustrated in the bottom, left corner of Fig. 6, which shows an increase in Italian purchases and a decrease in German purchases, with total purchases (green line) unchanged. The impact on Italian yields is striking. Reallocating purchases causes a large, persistent decrease in Italian yields (top panel of Fig. 8) of more than 15 basis points across most maturities, which is quite similar to the overall impact of the PEPP design. In contrast, the impact on German yields (not shown) is negligible, as we saw earlier in Fig. 7.

The conclusion from this exercise is that Core yields are driven by the overall quantity of purchases, not their distribution across jurisdictions. But for Periphery, country-specific purchases are crucial for yields, because decreasing free-floating default risk makes the market more willing to hold this risk. Again, we see that the actual impact on expected losses from default is tiny – the additional purchases of Italian debt decrease the expected losses from Italian default by less than one basis point (the “DP” component). But when part of this risk is taken off the market, arbitrageurs become much more willing to bear the remaining risk, lowering the credit risk premium by 20bp at intermediate maturities (the “CR” component). Hence, from the point of view of reducing average euro area yields, reallocating purchases from Core to Periphery makes the purchase program much more efficient. In other words, flexibility in allocation across countries is an important factor in explaining the effectiveness of the PEPP design.

The second panel of Figure 8 instead isolates the impact of flexibility in timing. It considers a frontloading scenario in which all purchases are realized in the first five months of the purchase program, as compared with the APP scenario in which the pace of purchases is constant through December 2020 (these two possible purchase paths are compared in the bottom, right panel of Fig. 6). This comparison relates to timing only, so it does not contemplate any deviation from capital keys; but note that
the frontloading in this exercise is substantially stronger than the frontloading in our baseline PEPP scenario. This frontloading causes a tiny decrease, on impact, in the German yield curve (not shown). For Italy (middle panel of Fig. 8) it causes a large decrease in short yields on impact, of more than 20bp for six-month maturities but at the same time, it causes a small increase in the longest yields. Frontloading implies an increase in the flow of purchases early in the program, but by the same token it implies a decrease in this flow later, and eventually causes the whole portfolio to mature earlier. Hence, the impact of frontloading over time is a sharp decrease in most yields at the beginning of the program, but a small increase later, when the portfolio matures. These future effects are priced into Italian ten-year yields from the very beginning.

We have seen that reallocation can have a big impact on peripheral yields, with negligible effects on Core, while frontloading trades off a large decrease in peripheral yields on short-term bonds with a smaller increase in peripheral yields on long bonds. But how do these policies interact? The last panel of Figure 8 addresses this question by comparing the individual effects of flexibility across jurisdictions, and flexibility over time, to their joint effect. The dotted line sums the impact of flexibility across jurisdictions only (redistributing purchases from Germany to Italy, away from their capital keys, as illustrated in the lower, left panel of Fig. 6) plus the impact of frontloading only (as illustrated in the lower, right panel of Fig. 6). The solid line instead shows the joint, nonlinear effect of a policy that combines reallocation across jurisdictions with frontloading over time. Hence, the difference between the solid line and the dashed line shows that reallocation and frontloading interact in a nonlinear way. In particular, combining reallocation with frontloading decreases short yields by up to 40 basis points, while the impact of the two policies separately sums to only 33 basis points for the same maturities. We conclude that flexibility across jurisdictions and flexibility across time are complements, though the degree of complementarity is rather small.

Finally, we consider an aspect of flexibility still available to the PEPP program. Without changing the structure of the initial purchases, the impact of the program might be increased by reinvesting maturing bonds after net purchases end. In Figure 9, we consider an extension of our baseline PEPP scenario that would reinvest all maturing bonds in 10-year bonds, for a period of five years, after which the whole portfolio is held to maturity. Note that since our baseline PEPP scenario assumed that purchases would be uniform across maturities, reinvesting in 10-year bonds serves to maintain this uniform distribution across maturities throughout the five-year reinvestment period.
Comparing the impact on Italian yields of reallocation of purchases across countries, and front-loading over time, separately and jointly. The dashed line shows the sum of the separate effects of reallocation and frontloading, while the solid line shows the combined impact of implementing reallocation and frontloading jointly.

Notes.

Figure 8: Effects of flexibility in allocation and timing
Figure 9: Impact of reinvestments

Notes.
Panels show the difference between a scenario in which all maturing bonds are reinvested in new purchases of 10Y bonds, for five years after the end of net purchases, and the baseline PEPP scenario without reinvestment.

Figure 9 reports the difference in the behavior of yields between the PEPP scenario with reinvestment, and the baseline PEPP scenario we analyzed earlier. Even though the profile of purchases over the net investment period is unchanged, committing to reinvestment has a large impact on yields from the time of announcement, especially for longer bonds. At the time of announcement, the program with reinvestment decreases the yield on German 10-year bonds by an additional five basis points, and that on Italian 10-year bonds by an additional seventy basis points. As before, the impact on Italian bonds goes mainly through the credit risk premium, since the reinvestment program decreases for several years the net supply of defaultable bonds that

\[ \text{\textsuperscript{21}} \text{For comparability with our earlier exercises, we assume that the whole time path of each scenario is known at the time of the initial PEPP announcement, i.e. March 2020.} \]
arbitrageurs must hold. The decrease in German and Italian yields is persistent, with
the maximum impact on 10-year bonds after four years, and the maximum impact on
Italian short-term bonds (almost a full percentage point) nine years down the line.
Hence, reinvestment still offers the ECB a long-lasting strategy to compress average
European yields, and to avoid fragmentation between Core and Periphery.

6 Conclusions

In this paper, we propose a micro-founded model of the term structure of sovereign
interest rates in a heterogeneous monetary union. We extend the Vayanos and Vila
(2020) term-structure model to a two-country monetary union setup in which one of the
two sovereign issuers (Periphery) has default risk, due to the possibility of rollover crises.
In addition to the standard and well-documented ’duration extraction channel’ of asset
purchases programs, our model features a ’sovereign risk extraction channel’, whereby
announcements of central bank asset purchases reduce both the expected amount of
defaultable bonds to be absorbed by the market and the default probability itself, thus
reducing the compensation required by investors to absorb default risk.

We apply our model to analyze the impact of the ECB’s pandemic emergency pur-
chase program (PEPP), announced in mid-March in a context of rising expected emis-
sions of euro area sovereign debt as a consequence of the Covid-19 crisis. We calibrate
the model to data on German and Italian yields, by targeting the average shape of
both countries’ sovereign yield curves in the pre-pandemic period and their change in
the two days following the PEPP announcement. We find that default risk extraction
is the most significant channel to explain the shape of the response of the Italian yield
curve to the PEPP announcement, much more so than duration extraction.

We then perform counterfactual simulations to evaluate how important the PEPP
flexibility was to explain its impact. Indeed, a key feature of the PEPP was the flexibility
in the distribution of purchase over time and across jurisdictions. We find that the
flexible allocation of purchases permitted by the PEPP program substantially enhanced
its impact. The PEPP announcement reduced Italian yields by around 80bp across
the yield curve, of which roughly 15bp can be attributed to the flexibility of PEPP, as
compared with an APP-like announcement of purchases at a constant rate and allocated
across countries according to the ECB’s capital key. We also show that the return
on riskless bonds is largely determined by the overall level of purchases – regardless
of its cross-country distribution. However, average euro-area yields depend strongly on how purchases are allocated, because reallocation towards peripheral bonds has a large impact on the yield of peripheral bonds but a negligible impact on that of core bonds.

References


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A Appendix: Modelling details

A.1 Sovereign default

To avoid default in the case of a rollover crisis, the government must impose emergency taxation, which it views as costly. We suppose that the government minimizes the expected discounted flow of emergency taxation, discounted at its subjective discount rate \( \hat{r} \):

\[
V^R_0[f_0(\cdot), f^{CB}_0(\cdot)] = \mathbb{E}_0 \left\{ \int_0^\infty e^{-(\hat{r} + \phi)t} \left( \Pi_t \underbrace{\text{Stream of emergency taxation}}_{\text{Loss after the crisis}} + \phi V_t[f_t(\cdot), f^{CB}_t(\cdot)] \right) dt \right\},
\]

where \( V_t[f_t(\cdot), f^{CB}_t(\cdot)] \) is the loss after the crisis and \( \phi \) is the Poisson rate parameter governing the duration of the crisis.

If a country defaults, a share \( 0 < \delta < 1 \) of its outstanding debt is repudiated, while the remainder is unaffected. Notice that, at the aggregate level, this is equivalent to assuming that the face value of each individual bond is reduced by \( \delta \). For simplicity, we assume that, after a default, the government replaces the defaulted bonds by new ones, with the same maturity profile. We assume that the gains related to this reissuance accrue to international organizations, such as the IMF, that may intervene in the case of a sovereign debt crisis. This makes default a direct transfer of wealth from investors to the international organizations, while from the defaulting government’s perspective, it alleviates the short-run costs of imposing emergency taxation. Furthermore, if a country defaults, we assume it suffers a random welfare loss \( \chi \) distributed according to a cdf \( \Phi \). The loss function in the case of default is given by

\[
V^D_0[f_0(\cdot), f^{CB}_0(\cdot)] = V_0[f_0(\cdot), f^{CB}_0(\cdot)] + \chi.
\]

The optimal decision to default at the beginning of a crisis will thus depend on \( \min[V^R_0, V^D_0] \). The continuation loss is given by

\[
V_0[f_0(\cdot), f^{CB}_0(\cdot)] = \mathbb{E}_0 \left\{ \int_0^\infty e^{-(\hat{r} + \eta)t} \eta \min[V^R_t, V^D_t] dt \right\}.
\]
If crises are low-probability events \((\eta \rightarrow 0)\),\(^{22}\) the continuation value is approximately zero, \(V_t \rightarrow 0\). Hence, if a rollover crisis arrives at time 0, the probability of default is the probability that repayment is more costly than default, that is:

\[
P(\text{default at time 0}|\text{crisis}) = P (V_0^R > V_0^D) \approx P (V_0^R > \chi) = \Phi (V_0^R). \quad (28)
\]

Combining the loss function \((27)\) with the budget constraint \((1)\), in the context of a rollover crisis, so that issuances \(\iota(\tau)\) must be zero, we get

\[
V_0^R = \int_0^\infty e^{-(\delta + \phi)t} E_0 [d_t + f_t (0) - \Gamma_t] dt.
\]

Thus, the loss in case of repayment depends on the seigniorage policy of the central bank during a rollover crisis. We assume that, in this case, the central bank pays out to the national government the income it receives from redemptions of the government’s bonds minus a fixed amount \(\bar{\Gamma}\). In other words, we assume a seigniorage rule of the form

\[
\Gamma_t = f_t^{CB} (0) - \bar{\Gamma}, \quad (29)
\]

so that payments from the government to the central bank to redeem maturing bonds are partially transferred back again to the government in the form of seigniorage. If short-term rates remain below or close to zero, the rule \((29)\) can be sustained as central bank liabilities are not remunerated. Hence, the central bank can transfer maturing bonds back to the treasury without any impact on the volume of reserves or its capital. In the case of positive interest rates, a rule like \((29)\) cannot avoid an increase in reserves during a crisis. The reserves associated with purchases of peripheral bonds follow

\[
\dot{D}_t = r_t D_t + \int P_t (\tau) \iota_t^{CB} (\tau) d\tau + \Gamma_t - f_t^{CB} (0), \quad (30)
\]

so in this case, \(\dot{D}_t = r_t D_t + \bar{\Gamma}\) during a crisis, as \(\iota_t^{CB} (\tau) = 0\) and \(\Gamma_t = f_t^{CB} (0) - \bar{\Gamma}\). This will lead to an expansion in the volume of reserves as long as \(\bar{\Gamma} < r_t D_t\).

\(^{22}\)Our estimate of the compound parameter \(\theta \equiv \eta \Phi'\) is slightly greater than one basis point per month, so the assumption that \(\eta\) is tiny is reasonable.
We can define the national central bank’s capital as

\[ K_t = \int \tilde{P}_t(\tau) f_t^{CB}(\tau) \, d\tau - D_t, \]

where, as in Del Negro & Sims (2015), \( \tilde{P}_t(\tau) \) is a "historical" price that changes only when gross purchases are positive (\( f_t^{CB}(\tau) > 0 \)). Capital then evolves as follows,

\[ \dot{K}_t = \int \left( \frac{d\tilde{P}_t(\tau)}{dt} f_t^{CB}(\tau) + \tilde{P}_t(\tau) \dot{f}_t^{CB}(\tau) \right) \, d\tau - \dot{D}_t. \]

During a rollover crisis, \( \frac{d\tilde{P}_t(\tau)}{dt} = 0 \) and \( \dot{D}_t = r_t D_t + \bar{\Gamma} \), so that

\[ \dot{K}_t = \int \left( \tilde{P}_t(\tau) \dot{f}_t^{CB}(\tau) \right) \, d\tau - r_t D_t + \Gamma. \]

Capital can thus decrease during a crisis, potentially falling below zero. This will depend on the maturity structure of the central bank assets, the volume of reserves, the path of real interest rates and the constant term \( \bar{\Gamma} \). Nonetheless, as discussed by Del Negro and Sims (2017) and Reis (201X), a central bank can operate with low or even negative capital.

Given this seigniorage rule, solving for the default probability is greatly simplified, because it depends on deficits and net bond issuance alone, without requiring us to track other state variables such as the quantity of reserves. Concretely, under our assumption that \( d_t, f_t \) and \( f_t^{CB} \) follow deterministic paths, the probability of default is

\[ \Phi_0 \equiv \mathbb{P}(\text{default at time 0|crisis}) = \Phi \left( \int_0^\infty e^{-(\bar{r} + \phi)t} \{ d_t + S_t(0) + \bar{\Gamma} \} \, dt \right). \quad (31) \]

where \( S_t(0) = f_t(0) - f_t^{CB}(0) \).

A.2 Model solution

Since preferred habitat demand is assumed to be an affine function of yield, equations (15) and (16) imply that the risk prices \( \lambda_t \) and \( \xi_t \) must be affine too. Hence, a solution
requires \( \lambda_t = \Lambda_t r_t + \bar{\lambda}_t \) and \( \xi_t = \Xi_t r_t + \bar{\xi}_t \), where

\[
\Lambda_t \equiv -\gamma \sigma^2 \int_0^\infty (\alpha (\tau) [A_t (\tau)]^2 + \alpha^* (\tau) [A_t^* (\tau)]^2) \, d\tau,
\]

\[
\bar{\lambda}_t \equiv \gamma \sigma^2 \int_0^\infty [(S_t (\tau) - h_t (\tau) - \alpha (\tau) C_t (\tau)) A_t (\tau) + (S_t^* (\tau) - h_t^* (\tau) - \alpha^* (\tau) C_t^* (\tau)) A_t^* (\tau)] \, d\tau,
\]

\[
\Xi_t \equiv -\gamma \psi_1 \delta^2 \int_0^\infty \alpha (\tau) A_t (\tau) \, d\tau,
\]

\[
\bar{\xi}_t \equiv \gamma \psi_1 \delta^2 \int_0^\infty (S_t (\tau) - h_t (\tau) - \alpha (\tau) C_t (\tau)) \, d\tau.
\]

With this notation, if we substitute \( \mu_t (\tau) \) and \( \mu_t^* (\tau) \) from (10)-(11) into (12)-(13), the first-order conditions on the arbitrageurs’ portfolio weights are:

\[
0 = -\left( \frac{\partial A_t}{\partial \tau} - \frac{\partial A_t}{\partial t} \right) r_t - \left( \frac{\partial C_t}{\partial \tau} - \frac{\partial C_t}{\partial t} \right) + A_t (\tau) \kappa (\bar{r} - r_t) - \frac{1}{2} \sigma^2 [A_t (\tau)]^2 + r_t \\
+ A_t (\tau) (\Lambda_t r_t + \bar{\lambda}_t) + \psi_1 \delta + (\Xi_t r_t + \bar{\xi}_t),
\]

and

\[
0 = -\left( \frac{\partial A_t^*}{\partial \tau} - \frac{\partial A_t^*}{\partial t} \right) r_t - \left( \frac{\partial C_t^*}{\partial \tau} - \frac{\partial C_t^*}{\partial t} \right) + A_t^* (\tau) \kappa (\bar{r} - r_t) - \frac{1}{2} \sigma^2 [A_t^* (\tau)]^2 + r_t \\
+ A_t^* (\tau) (\Lambda_t r_t + \bar{\lambda}_t).
\]

Substituting in \( \Lambda_t, \bar{\lambda}_t, \Xi_t, \) and \( \bar{\xi}_t, \) and grouping the terms with and without \( r, \) we get

\[
0 = -\frac{\partial A_t}{\partial \tau} + \frac{\partial A_t}{\partial t} - A_t (\tau) \kappa + 1 + \Lambda_t A_t (\tau) + \Xi_t. \tag{32}
\]

\[
0 = -\frac{\partial C_t}{\partial \tau} + \frac{\partial C_t}{\partial t} + A_t (\tau) \kappa \bar{r} - \frac{1}{2} \sigma^2 [A_t (\tau)]^2 + \bar{\lambda}_t A_t (\tau) + \psi_1 \delta + \bar{\xi}_t. \tag{33}
\]

\[
0 = -\frac{\partial A_t^*}{\partial \tau} + \frac{\partial A_t^*}{\partial t} - A_t^* (\tau) \kappa + 1 + \Lambda_t A_t^* (\tau) \tag{34}
\]

\[
0 = -\frac{\partial C_t^*}{\partial \tau} + \frac{\partial C_t^*}{\partial t} + A_t^* (\tau) \kappa \bar{r} - \frac{1}{2} \sigma^2 [A_t^* (\tau)]^2 + \bar{\lambda}_t A_t^* (\tau). \tag{35}
\]

This provides a system of PDEs to determine functions \((A_t (\tau), C_t (\tau))\) and \((A_t^* (\tau), C_t^* (\tau))\), verifying our guess that the bond price is an affine function of \( r_t \).
A.3 Derivation of analytical results in Section 3

Here we provide the derivation of the different results discussed in Section 3. Our starting point is the system of equations (32-35) presented in Appendix A.2. In steady state, it simplifies to

\[ 0 = -\frac{\partial A}{\partial \tau} - A(\tau) \kappa + 1 + \Lambda A(\tau) + \Xi. \] (36)

\[ 0 = -\frac{\partial C}{\partial \tau} + A(\tau) \kappa \bar{r} - \frac{1}{2} \sigma^2 [A(\tau)]^2 + \bar{\lambda} A(\tau) + \psi \delta + \bar{\xi}. \] (37)

\[ 0 = -\frac{\partial A^*}{\partial \tau} - A^*(\tau) \kappa + 1 + \Lambda A^*(\tau) \] (38)

\[ 0 = -\frac{\partial C^*}{\partial \tau} + A^*(\tau) \kappa \bar{r} - \frac{1}{2} \sigma^2 [A^*(\tau)]^2 + \bar{\lambda} A^*(\tau), \] (39)

where we have suppressed the time index as functions are time-invariant. Equations (36) and (38) can be solved as

\[ A^*(\tau) = \frac{1 - e^{-\hat{\kappa} \tau}}{\hat{\kappa}}, \quad A(\tau) = \frac{1 + \Xi - e^{-\hat{\kappa} \tau}}{\hat{\kappa}}, \] (40)

where

\[ \hat{\kappa} = \kappa - \Lambda = \kappa + \gamma \sigma^2 \int_0^\infty \left( \alpha(\tau) \left( \frac{1 + \Xi - e^{-\hat{\kappa} \tau}}{\hat{\kappa}} \right)^2 + \alpha^*(\tau) \left( \frac{1 - e^{-\hat{\kappa} \tau}}{\hat{\kappa}} \right)^2 \right) d\tau. \]

and, integrating equations (37) and (39), we get

\[ C^*(\tau) = \int_0^\tau \left[ A^*(\tau) \kappa \bar{r} - \frac{1}{2} \sigma^2 [A^*(\tau)]^2 + \bar{\lambda} A^*(\tau) \right] d\tau, \]

\[ C(\tau) = (\psi \delta + \bar{\xi}) \tau + \int_0^\tau \left[ A(\tau) \kappa \bar{r} - \frac{1}{2} \sigma^2 [A(\tau)]^2 + \bar{\lambda} A(\tau) \right] d\tau. \]

A.4 Numerical algorithm

A.4.1 Finite-difference computation of the stochastic steady state

The stochastic steady state of our model must satisfy the system of ODEs (36)-(39). These can be solved by a finite difference method.\textsuperscript{23} To do so, we consider a grid of

\textsuperscript{23}We have defined and computed both continuous-time and discrete-time versions of the model. The discrete time version is described in the next section. Numerical simulations of both versions give the
maturities \((\tau_1, \ldots, \tau_I)\) with \(\tau_0 = 0\) and constant step size \(\Delta \tau\), so that \(\tau_i \equiv \tau(i) = i \Delta \tau\). Define

\[
A_i = A(\tau_i), A_i^* = A^*(\tau_i), C_i = C(\tau_i), C_i^* = C^*(\tau_i),
\]
\[
S_i = S(\tau_i), S_i^* = S^*(\tau_i), \alpha_i = \alpha(\tau_i), \alpha_i^* = \alpha^*(\tau_i),
\]
\[
h = h(\tau_i), h_i^* = h^*(\tau_i).
\]

The boundary conditions are \(A_0 = A(0) = 0\) and \(C_0 = C(0) = 0\) because an instantaneous bond trades at par. We begin with a guess of \(A_i^n, A_i^{n*}\), with \(n = 0\). For instance, we can begin with \(A_i^n = A_i^{n*} = \tau_i\) and \(C_i^n = C_i^{n*} = 0\). Then, considering a backward finite-difference approximation \(\frac{\partial A_{i+1}(\tau(i))}{\partial \tau} \approx \frac{A_{i+1} - A_i}{\Delta \tau}\), and likewise for the other unknown functions, we approximate the ODEs as:

\[
\frac{A_i^{n+1} - A_i^{n+1}}{\Delta \tau} = A_i^{n+1}A_i^n - A_i^{n+1}\kappa + 1 + \Xi^n,
\]
\[
\frac{C_i^{n+1} - C_i^{n+1}}{\Delta \tau} = A_i^{n+1}\bar{\lambda}^n + A_i^{n+1}\kappa\bar{r} - \frac{1}{2}\sigma^2 [A_i^{n+1}]^2 + \psi_{ss}\delta + \bar{\xi}^n,
\]
\[
\frac{A_i^{(n+1)*} - A_i^{(n+1)*}}{\Delta \tau} = A_i^{(n+1)*}A_i^n - A_i^{(n+1)*}\kappa + 1,
\]
\[
\frac{C_i^{(n+1)*} - C_i^{(n+1)*}}{\Delta \tau} = A_i^{(n+1)*}\bar{\lambda}^n + A_i^{(n+1)*}\kappa\bar{r} - \frac{1}{2}\sigma^2 [A_i^{(n+1)*}]^2,
\]

where

\[
A^n = -\gamma\sigma^2 \sum_{i=1}^I \left( \alpha_i [A_i^n]^2 + \alpha_i^* [A_i^{n*}]^2 \right) \Delta \tau,
\]
\[
\bar{\lambda}^n = \gamma\sigma^2 \sum_{i=1}^I \left[ (S_i - h_i - \alpha_i C_i^n) A_i^n + (S_i^* - h_i^* - \alpha_i^* C_i^{n*}) A_i^{n*} \right] \Delta \tau,
\]
\[
\Xi^n = -\gamma\psi_{ss}\delta^2 \sum_{i=1}^I \alpha_i A_i^n \Delta \tau.
\]
\[
\bar{\xi}^n = \gamma\psi_{ss}\delta^2 \sum_{i=1}^I \left[ (S_i - h_i - \alpha_i C_i^n) \right] \Delta \tau.
\]

same results.
In matrix form, this amounts to

\[
\begin{bmatrix}
\frac{1}{\Delta \tau} - \Lambda^n + \kappa & 0 & 0 & \cdots & 0 \\
-\frac{1}{\Delta \tau} & \frac{1}{\Delta \tau} - \Lambda^n + \kappa & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \frac{1}{\Delta \tau} - \Lambda^n + \kappa \\
0 & 0 & \cdots & \cdots & -\frac{1}{\Delta \tau}
\end{bmatrix}
\begin{bmatrix}
F^n \\
\Lambda^{n+1}
\end{bmatrix}
= \begin{bmatrix}
1 + \Xi^n \\
\Lambda^{n+1} \\
\Lambda^{n+1}_I \\
\cdots \\
\Lambda^{n+1}_I \\
1 + \Xi^n
\end{bmatrix} \tag{41}
\]

\[
\begin{bmatrix}
\frac{1}{\Delta \tau} - \Lambda^n + \kappa & 0 & 0 & \cdots & 0 \\
-\frac{1}{\Delta \tau} & \frac{1}{\Delta \tau} - \Lambda^n + \kappa & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \frac{1}{\Delta \tau} - \Lambda^n + \kappa \\
0 & 0 & \cdots & \cdots & -\frac{1}{\Delta \tau}
\end{bmatrix}
\begin{bmatrix}
C^{n+1} \\
\cdots \\
I^n \\
C_{I-1}^{n+1} \\
C_{I-1}^{n+1}
\end{bmatrix}
= \begin{bmatrix}
A_1^{n+1} \\
A_2^{n+1} \\
\cdots \\
A_{I-1}^{n+1} \\
A_I^{n+1}
\end{bmatrix}
\tag{42}
\]

\[
\begin{bmatrix}
\frac{1}{\Delta \tau} & 0 & 0 & \cdots & 0 \\
-\frac{1}{\Delta \tau} & \frac{1}{\Delta \tau} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \frac{1}{\Delta \tau}
\end{bmatrix}
\begin{bmatrix}
G \\
\cdots \\
\cdots \\
\cdots \\
\cdots
\end{bmatrix}
= \begin{bmatrix}
A_1^{n+1} \bar{\lambda}^n + A_1^{n+1} \kappa \bar{r} - \frac{1}{2} \sigma^2 [A_1^{n+1}]^2 + \psi_{ss} \delta + \delta^n \\
A_2^{n+1} \bar{\lambda}^n + A_2^{n+1} \kappa \bar{r} - \frac{1}{2} \sigma^2 [A_2^{n+1}]^2 + \psi_{ss} \delta + \delta^n \\
\vdots \\
A_{I-1}^{n+1} \bar{\lambda}^n + A_{I-1}^{n+1} \kappa \bar{r} - \frac{1}{2} \sigma^2 [A_{I-1}^{n+1}]^2 + \psi_{ss} \delta + \delta^n \\
A_I^{n+1} \bar{\lambda}^n + A_I^{n+1} \kappa \bar{r} - \frac{1}{2} \sigma^2 [A_I^{n+1}]^2 + \psi_{ss} \delta + \delta^n
\end{bmatrix} \tag{43}
\]

where we have already applied the boundary conditions.

The idea is to solve equations (41) and (42) iteratively from the initial guess, updating \( \Lambda^n \) and \( \Xi^n \) and at each step, and then calculate \( \bar{\lambda}^n \) and \( \delta^n \) in order to solve (43)
and (44) in a single step.

A.4.2 Computation of the dynamics

To compute the dynamics, consider a distant terminal time $T$ at which the model has converged to its steady state. We solve the PDEs (32)-(35) backwards from time $T$ with time steps of size $\Delta t \equiv \Delta \tau$, so that backwards induction step $n$ refers to calendar time $t(n) \equiv T - n\Delta \tau$. Using the fact that $A_i^{n+1} - A_i^n \approx -\frac{\partial A^n(\tau(i))}{\partial \tau}\Delta \tau$, the PDEs can be discretized as follows:

\[
\begin{align*}
\frac{A_i^{n+1} - A_i^n}{\Delta \tau} + F^n A^n &= f^n, \\
\frac{A_i^{(n+1)*} - A_i^{n*}}{\Delta \tau} + F^n A^{n*} &= f^*, \\
\frac{C_i^{n+1} - C_i^n}{\Delta \tau} + GC^n &= g^n, \\
\frac{C_i^{(n+1)*} - C_i^{n*}}{\Delta \tau} + GC^{n*} &= g^{n*}.
\end{align*}
\]

Matrices $F^n, G, f^n, f^*, g^n, g^{n*}$ are defined as before, except that we calculate $\Lambda_t, \Xi_t, \bar{\lambda}_t,$ and $\bar{\xi}_t$ under time-varying conditions. In particular, we evaluate them conditional on the net bond supply $S_t(\tau)$ and default probability $\psi_t$ at time $t = t(n)$.

A.5 Discrete time representation

It is straightforward to derive and compute a discrete-time framework that is equivalent to our continuous-time model. In discrete time, we write the price of a bond with a maturity of $i$ periods, issued by jurisdiction $j \in \{P, C\}$, where $P$ stands for “periphery” and $C$ for “core”, as $P_i^{j,t} = \exp(p_i^{j,t}) = \exp\left(-A_i^{j,t}\bar{r}_t - C_i^{j,t}\right)$. Let the rate on reserves follow $r_{t+1} = \rho r_t + (1 - \rho)\bar{r} + \sigma \varepsilon_{t+1}$, where $\varepsilon_{t+1} \sim N(0, 1)$. If arbitrageurs maximize a mean-variance utility function over the increase of their wealth, then if the time period

\footnote{Inspecting the definitions of the matrices above, we can see that this algorithm calculates equilibrium objects at time $t(n) - \Delta \tau$ using the risk prices $\lambda_{t(n)}$ and $\xi_{t(n)}$ from time $t(n)$. It would therefore be incorrect to apply this algorithm with a large time step $\Delta \tau$, but in the limit as $\Delta \tau \rightarrow 0$, it gives the correct solution of the continuous-time PDE.}
is sufficiently short, their optimization problem can be approximated as follows:\textsuperscript{25}

\[
\max_{\{X_{i,t}^j\}} \left( W_t - \sum_{i=1}^{I} \sum_{j \in \{P,C\}} X_{i,t}^j \right) r_t + \sum_{i=1}^{I} \sum_{j \in \{P,C\}} X_{i,t}^j \left( -C_{i-1,t+1}^j - A_{i-1,t+1}^j ((1 - \rho)\bar{r} + \rho r_t) + C_{i,t}^j + A_{i,t}^j r_t + \frac{\sigma^2}{2} (A_{i-1,t+1}^j)^2 \right) - \delta \psi_t^j
\]

where \( \psi_t^C = 0 \) denotes the core default probability, and \( \psi_t^P = \psi_t \) is the peripheral default probability, given by (5). Hence, the first-order condition on the investment \( X_{i,t}^j \) in bonds of maturity \( i \) from jurisdiction \( j \) is

\[
r_t = - (C_{i-1,t+1}^j + A_{i-1,t+1}^j ((1 - \rho)\bar{r} + \rho r_t)) + (C_{i,t}^j + A_{i,t}^j r_t) + \frac{\sigma^2}{2} (A_{i-1,t+1}^j)^2 - \delta \psi_t^j - A_{i-1,t+1}^j \lambda_t - \xi_t^j,
\]

where

\[
\lambda_t = \gamma \sigma^2 \left[ \sum_{i=1}^{I} \sum_{j \in \{P,C\}} X_{i,t}^j A_{i-1,t+1}^j \right],
\]

\[
\xi_t^j = \gamma \psi_t^j \delta^2 \sum_{i=1}^{I} X_{i,t}^j.
\]

Note that since \( A_{0,t}^j = C_{0,t}^j = 0 \), the first-order condition for holdings of one-period core bonds is simply

\[
r_t = y_{1,t}^C = C_{i,t}^j + A_{i,t}^j r_t,
\]

and the FOC for longer bonds can be interpreted as

\[
p_{i,t}^j = -r_t + E_t p_{i-1,t+1}^j + \frac{1}{2} Var_t p_{i-1,t+1}^j - A_{i-1,t+1}^j \lambda_t - \delta \psi_t^j - \xi_t^j,
\]

or equivalently

\[
P_{i,t}^j = \exp \left( -r_t - A_{i-1,t+1}^j \lambda_t - \delta \psi_t^j - \xi_t^j \right) E_t p_{i-1,t+1}^j.
\]

We now apply the market clearing condition \( X_{i,t}^j = S_{i,t}^j - Z_{i,t}^j \), where preferred-habitat demand is \( Z_{i,t}^j = h_{i,t}^j - \alpha_{i,t}^j p_{i,t}^j \), and we write the risk compensation terms in

\textsuperscript{25} See Hamilton and Wu (2012) for details.
affine form as \( \lambda_t = \Lambda_t r_t + \bar{\lambda}_t \) and \( \xi_t^P = \Xi_t^P r_t + \bar{\xi}_t^P \), with \( \xi_t^C = \Xi_t^C = \bar{\xi}_t^C = 0 \). Then, the first-order conditions imply the following restrictions on the affine pricing coefficients:

\[
A^j_{i,t} = A^j_{i,t-1,t+1} (\rho + \Lambda_t) + \Xi^j_t, \tag{45}
\]

\[
C^j_{i,t} = C^j_{i,t-1,t+1} - \frac{1}{2} \sigma^2 A^j_{i-1,t+1} + A^j_{i-1,t+1} ((1 - \rho) \bar{\lambda}_t + \bar{\xi}_t^P) + \delta \psi_t + \bar{\xi}_t^P, \tag{46}
\]

where

\[
\Lambda_t = -\gamma \sigma^2 \sum_{i=2}^I \sum_{j \in \{P,C\}} A^j_{i-1,t+1} (\alpha^j_i A^j_i), \tag{47}
\]

\[
\bar{\lambda}_t = \gamma \sigma^2 \sum_{i=2}^I \sum_{j \in \{P,C\}} A^j_{i-1,t+1} (S^j_{i,t} - h^j_{i,t} - \alpha^j_i C^j_i), \tag{48}
\]

\[
\Xi_t^P = -\gamma \delta^2 \psi_t P \sum_{i=1}^I (\alpha^P_i A^P_i), \tag{49}
\]

\[
\bar{\xi}_t^P = \gamma \delta^2 \psi_t P \sum_{i=1}^I (S^P_{i,t} - h^P_{i,t} - \alpha^P_i C^P_i). \tag{50}
\]

These difference equations can be solved by backwards induction, starting from a distant time \( T \) at which we assume that the pricing functions are known, bearing in mind that \( A^j_{0,t} = C^j_{0,t} = 0 \) for all \( j \) and \( t \). To ensure a correct solution of the discrete-time model, we can apply a fixed-point calculation at each time step:

1. Guess \( A^j_{i,t} = A^j_{i,t+1} \) and \( C^j_{i,t} = C^j_{i,t+1} \).
2. Calculate \( \Lambda_t, \Xi_t^P, \bar{\lambda}_t, \) and \( \bar{\xi}_t^P \) from (47)-(50).
3. Update \( A^j_{i,t} = A^j_{i,t+1} \) and \( C^j_{i,t} = C^j_{i,t+1} \) using (45)-(46).
4. Iterate to convergence.

Once the time \( t \) equilibrium has been calculated, we can step backwards to calculate the time \( t - 1 \) equilibrium by the same method.