Decomposing Optimal Debt Maturity*

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Abstract

We characterize optimal sovereign debt maturity when a government issues finite-life bonds with liquidity costs and default risk. We derive a value-gap rule decomposing maturity choice into four forces: hedging, self-insurance, credit risk, and dilution. A risky steady state makes the infinite-dimensional problem tractable and permits transitional dynamics. Calibrated to Spain (2002–2018) with auction-based liquidity costs and observed curves, hedging dominates while credit-risk effects are minor. The model recommends longer maturities; a permanent 1pp long-rate increase cuts issuance by 0.5% of GDP; a 50-year bond shifts issuance long without raising rollover risk.

Keywords: Debt maturity; Default; Liquidity costs;

JEL classification: F34, F41, G11, E62

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1. Introduction

Over the last forty years, many economies have increased public debt to unprecedented levels. At such levels, the interest burden and the ability to service debt become highly sensitive to the maturity structure. Optimal maturity management is therefore a first-order macroeconomic policy problem—and it is likely to be even more central in a post-COVID-19 inflationary environment if real interest rates remain high.

The theories of public finance and international finance guide treasuries in designing maturity strategies. Existing models emphasize several forces behind maturity choice—hedging, self-insurance, credit risk, and incentives related to debt dilution—but typically study only a subset, and often in settings where these forces are tightly intertwined. As a result, it is difficult to disentangle how optimal prescriptions depend on each force separately and to assess their quantitative importance.

This paper develops an approach that isolates and quantifies these forces within a unified framework. We study a small open economy in which the government issues finite-life bonds across maturities to risk-neutral foreign investors in order to maximize household welfare. The economy is hit by shocks to income and to the foreign risk-free rate, and the government may default strategically upon a large shock, in the spirit of Eaton and Gersovitz (1981) and Arellano (2008). The government can commit to a debt program prior to the shock. Issuance faces liquidity costs: auction prices decline with issuance volume as in Faraglia et al. (2018) and Bigio et al. (2022).

Optimal debt issuance at each maturity is proportional to a *value gap*—the wedge between the market price (risk-neutral valuation) and the domestic valuation (government's shadow price)—and inversely proportional to the slope of liquidity costs. This mapping lets us decompose the maturity profile into four forces: (i) (partial) hedging, (ii) self-insurance, (iii) credit risk, and (iv) incentives to avoid debt dilution—captured by a "revenue echo" effect term. Absent liquidity costs, the government would *hedge* by choosing a portfolio that generates capital gains in adverse interest-rate states (Duffie and Huang, 1985; Angeletos, 2002; Buera and Nicolini, 2004); with liquidity costs, perfect hedging is infeasible and expenditures can jump. Expected jumps in prices and valuations then tilt ex-ante issuance toward maturities that become relatively "cheaper" for the government. A jump in marginal utility after a shock effectively lowers the

domestic discount rate, yielding *self-insurance*—lower issuance and longer maturity. *Credit risk* affects both prices and valuations through default probabilities and, for a risk-averse government, partially substitutes for self-insurance. Finally, a marginal issuance increases default risk before the bond expires, which depresses auction revenues on other issuances; this *revenue echo* appears in domestic valuations but not in market prices, pushing toward less debt and longer average maturity.

Because the state is the entire, time-varying maturity distribution, characterizing equilibrium confronts the same computational challenges as heterogeneous-agent models with aggregate shocks (Krusell and Smith, 1998). To obtain sharp predictions, we introduce a *risky steady state* (RSS): the asymptotic pre-shock limit when a one-time disaster shock is anticipated but has not yet materialized. The RSS yields closed-form expressions for prices and valuations, permits a clean force decomposition, and can be used to study small perturbations and transitional dynamics. Unlike perturbation methods around deterministic steady states with infinite-dimensional states (Reiter, 2009; Ahn et al., 2017; Boppart et al., 2018), our approach analyzes perturbations around a stochastic steady state in which aggregate risk is anticipated.

We apply the framework to Spain. We calibrate to (i) the maturities Spain issues (3, 5, 10, 15, and 30 years),[1] (ii) liquidity costs estimated from auction data, (iii) the Spanish yield curve—including the average default premium—over 2002–2018, and (iv) average issuance relative to GDP. The model replicates the hump-shaped issuance profile peaking at 10 years and recommends somewhat more issuance at 15–30 years and less at 3–10 years. The force decomposition shows that hedging against persistent interest-rate risk is quantitatively dominant, while credit-risk effects are small; the revenue echo and self-insurance restrain short-end issuance.

We then study a transition to a higher-rate environment in which the long-term real rate rises by 1 pp, and we assess the introduction of a 50-year bond. In the high-rate scenario, optimal policy reduces gross issuance by roughly 0.5% of GDP with little change in average maturity, implying lower long-run consumption due to the higher interest burden. If a 50-year bond is available, optimal policy issues amounts compa-

[1] We disregard bills—bonds of less than 1.5 years—because they have different features (they are zero-coupon, for instance) than medium- and long-term bonds, and their issuance appears to follow a different rationale, according to practitioners we have contacted.

rable to those at 30 years; the initial issuance boom implies a larger principal 50 years later and a temporary issuance dip, with issuance stabilizing at a higher level in the very long run. These experiments underscore the importance of a long-horizon perspective in debt-management policy.

Related literature. Several strands of the literature study sovereign debt management, each emphasizing different forces.

In public finance, anticipated risk motivates hedging and self-insurance. Lucas and Stokey (1983) shows that with state-contingent securities, the government should smooth taxes in a stochastic sense. Angeletos (2002) demonstrates that a government can implement the Lucas and Stokey (1983) allocation by hedging with a menu of fixed-income bonds, and Buera and Nicolini (2004) notes that, given observed bond-price volatility, implementing that allocation requires large positions. Barro (2003) studies a special case with deterministic income but stochastic discount rates, showing that issuing perpetuities minimizes costly rollovers. Related prescriptions arise in international finance, where rollover risk at short maturities can trigger default (Calvo, 1988; Cole and Kehoe, 2000), a mechanism explored quantitatively by Bocola and Dovis (2018). Even under short-sale constraints, long-term debt can hedge fiscal shocks (Lustig et al., 2008). In our setting, liquidity costs preclude perfect hedging, but partial hedging remains central.

Departing from complete markets, Aiyagari et al. (2002) analyzes the Lucas and Stokey (1983) problem with a single maturity, showing that self-insurance pushes toward lower debt. Bhandari et al. (2017) study how self-insurance shapes optimal maturity in an open economy with distortionary taxes.

The sovereign-default literature (e.g., Eaton and Gersovitz, 1981; Arellano, 2008; Aguiar and Amador, 2013) highlights dilution and maturity choice. Bulow and Rogoff (1988) emphasize that long-term debt is prone to *debt dilution*: new issuance fails to internalize the higher default premia on outstanding debt. Quantitative work studies the positive and normative implications of dilution with long-term bonds (Hatchondo and Martinez, 2009; Chatterjee and Eyigungor, 2012, 2015), as well as optimal maturity when both short- and long-term debt are available (Arellano and Ramanarayanan, 2012; Hatchondo et al., 2016). Closest to us, Aguiar et al. (219) analyzes optimal matu-

rity with multiple bonds and default and finds that issuing only at the shortest maturity is optimal; our results show that once liquidity costs are introduced, that conclusion can reverse.

Liquidity costs are central in our framework and are well documented in asset markets.[2] We build on Bigio et al. (2022), which exploits institutional features of Spain's primary and secondary markets to estimate maturity-specific liquidity costs that we use here, and develop an optimal debt-management theory with such costs. While their analysis focuses on small perturbations around a deterministic steady state without default, we present results that allow for anticipated aggregate risk and default and study how these interact with liquidity costs. Faraglia et al. (2018) also studies optimal maturity with finite-life bonds and liquidity costs, emphasizing their role in reconciling practice and theory.

Our contribution is to provide an analytic and quantitative *decomposition* of maturity choice into (partial) hedging, self-insurance, credit risk, and dilution incentives (the "revenue echo"), using a value-gap characterization and a risky steady state that makes the infinite-dimensional problem tractable. We see our paper as presenting a new tool for the optimal debt-management toolkit.

2. Model

2.1 Environment

Time is continuous, $t \in [0, \infty]$. A small open economy (SOE) is populated by a representative household with preferences over consumption paths, $\{c_t\}_{t\geq 0}$, given by

$$\mathcal{V}_0 = \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} U(c_t)\, dt \right],$$

where $\rho \in (0,1)$ is a discount rate and instantaneous utility $U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. There is a single freely traded good.

[2] See, among others, Cammack (1991); Spindt and Stolz (1992); Duffee (1996); Fleming (2002); Green (2004); Fleming and Rosenberg (2008); Pasquariello and Vega (2009); Krishnamurthy and Vissing-Jorgensen (2012); Pelizzon et al. (2016); Breedon (2018) and the surveys by Vayanos and Wang (2013) and Duffie (2010).

The government decides consumption on behalf of its citizens and issues bonds that differ by their time to maturity, $\tau \in [0,T]$, where T is an exogenous maximum maturity. Bonds pay a constant coupon δ until maturity. The government repays the principal, normalized to one, once the bond matures at $\tau = 0$. The outstanding stock of bonds with maturity τ at date t is denoted by $f_t(\tau) : [0,T] \to \mathbb{R}$. We refer to f_t as the *debt profile*. The debt profile satisfies a partial differential equation (PDE):

$$\frac{\partial f}{\partial t} = \iota_t(\tau) + \frac{\partial f}{\partial \tau}, \tag{1}$$

with boundary condition $f_t(T) = 0$ and, $f_0(\tau)$, given. This law of motion captures that, at time t, the change in the quantity of bonds of maturity τ, $\partial f/\partial t$, equals the issuance of bonds at that maturity, $\iota_t(\tau)$, plus the net flow of bonds, $\partial f/\partial \tau$. The net flow reflects that maturity decreases as bonds approach maturity. When negative, issuances represent purchases.

The government's instantaneous budget constraint is:

$$c_t = y_t - f_t(0) + \int_0^T \left[q_t(\tau,\iota)\iota_t(\tau) - \delta f_t(\tau)\right] d\tau, \tag{2}$$

where y_t is income, $-f_t(0)$ principal repayments, $-\delta\int_0^T f d\tau$ coupon payments, and $\int_0^T q\iota d\tau$ are the auction revenues from all issuances. The price of a τ-bond at date t is $q_t(\tau,\iota)$, a function of the auctioned amount. Following Faraglia et al. (2018) and Bigio et al. (2022), we consider linear liquidity costs:

$$\underbrace{q_t(\tau,\iota)}_{\text{selling price}} = \underbrace{\psi_t(\tau)}_{\text{market price}} - \underbrace{\frac{1}{2}\bar{\lambda}\psi_t(\tau)\cdot\iota}_{\text{liquidity costs}}, \tag{3}$$

where $\bar{\lambda} > 0$ is a parameter the captures the *price impact* per auction. This function induces a wedge between the auction price and the secondary market price of bonds. This wedge is proportional to the issuance amount of debt issued at a specific maturity per unit of time. Faraglia et al. (2018) argue that liquidity costs are important to explain actual debt management practices. Bigio et al. (2022) provides empirical evidence and presents an inventory model of the primary dealers in the debt market that rationalizes

this functional form.

2.2 Risk and default

We model risk as a single random jump (a single shock event). In particular, we let the exogenous state, $X_t = [y_t, r_t^*]$ denote the processes for income and the international risk-free rate further described below. These processes follow a compound Poisson process with intensity ϕ.

We introduce notation to distinguish pre- from post-jump variables: Denote by t^o the (random) date of a jump event. For any variable, we denote its pre-jump path by $\hat{x}_t$, and use x_t, without the hat symbol, to express its post-jump path.

Upon a jump, a new value of X is drawn from a distribution $G(\cdot|\hat{X}_{t^o})$ with convex and compact support. We denote the conditional expectation over G by $\mathbb{E}_t^X[\cdot]$. We define by $J_t^X[x_t] \equiv x_t/\hat{x}_t$ as the *jump* in x_t, conditional on the realization of X_t. [3] After the jump, x_t follows an exponential mean-reverting process:

$$\dot{x}_t = -\alpha^x(x_t - x_{ss})$$

where α^x is the reversion rate and x_{ss} the steady state value of x_t. There are no further jumps thereafter and X_t is the initial condition of this process.

The assumption of a single shock merits discussion: First, it is a "disaster shock"—a shock with low-probability probability of occurrence but high-impact. These types of shocks are shown to be relevant for macroeconomics and asset pricing, as argued in Barro (2006) and Gabaix (2012). Second, whereas shocks that occur only once are not a common feature in models, note that a one-time disaster shock well approximates recurrent disaster shocks. The reason is that any expected value after a second shock will be approximately discounted by ϕ^2.[4] Hence, if ϕ is small, this term will be of second-order compared to the effects of the first shock. This feature implies that the dynamics of a problem are predominantly governed by the effect of the first shock.[5] Most impor-

[3] Note that G can be conditioned on $\hat{X}_{t^o}$. Formally, X_t is right-continuous. If the jump occurs at t^o, the left-limit $\hat{X}_{t^o} \equiv \lim_{s\uparrow t^o} \hat{X}_s$ jumps to some new $X_{t^o} \sim G(\cdot; \hat{X}_{t^o})$.

[4] As Poisson shocks enter the Bellman equation multiplied by $\phi e^{-(\rho+\phi)t}$, the composition is discounted by $\phi^2 e^{-2(\rho+\phi)t}$,

[5] Furthermore, obtaining analytic solutions with recurrent shocks is not possible. The reason is that

tantly, this single-event approximation is key to producing the analytic results.

In addition to income and interest-rate risk, the model allows for default. In particular, the government can opt to default upon the arrival of a jump event. In that case, it defaults on all bonds and is barred from international markets thereafter.

The government's autarky value is

$$V_{t^o}^D \equiv V^D(X_{t^o}) \equiv \int_{t^o}^{\infty} e^{-\rho(s-t^o)} U\left((1-\kappa)y\left(s\right)\right) ds + \varepsilon. \tag{4}$$

The autarky value is the discounted value of the fraction $1-\kappa$ of the revenues without default, plus a zero-mean random variable, ε. The variable ε captures the randomness around the default decision. We denote by $\Theta(\cdot)$ and $\theta(\cdot)$ the CDF and PDF of $V_{t^o}^D$, respectively.[6] The government defaults if its autarky value exceeds the value of continuing to service debt, $V_{t^o} \equiv V\left[\hat{f}_{t^o}\left(\cdot\right), X_{t^o}\right]$. The latter is the perfect-foresight value of starting with an initial debt profile, $\hat{f}_{t^o}\left(\cdot\right)$. Thus, the default probability is $\mathbb{P}\left\{V_t^D > V_t\right\} = 1 - \Theta\left(V_t\right)$. To ease notation, we use $\Theta_t \equiv \Theta\left(V_t\right)$ and $\theta_t \equiv \theta\left(V_t\right)$.

2.3 International investors

The market price of a bond of maturity $\hat{\psi}$ is *arbitrage-free*:

$$\begin{aligned}\hat{\psi}_t(\tau) &= \underbrace{e^{-\int_t^{t+\tau}(\hat{r}^*(u)+\phi)du}}_{\text{principal repayment}} + \underbrace{\int_t^{t+\tau} e^{-\int_t^s(\hat{r}^*(u)+\phi)du}\delta ds}_{\text{coupons}} \\ &+ \underbrace{\phi\int_t^{t+\tau} e^{-\int_t^s(\hat{r}^*(u)+\phi)du}\mathbb{E}_s^X\left[\Theta\left(V\left[\hat{f}\left(\cdot,t\right)\right]\right)\psi\left(\tau-s,t+s\right)\right]ds}_{\text{expected discounted value of bond conditional on shock}},\end{aligned} \tag{5}$$

where $\hat{r_t}^*$ is the international risk-free rate and

$$\psi_t(\tau) = \underbrace{e^{-\int_t^{t+\tau} r_u^* du}}_{\text{principal repayment}} + \underbrace{\delta\int_t^{t+\tau} e^{-\int_t^s r_u^* du} ds}_{\text{coupons}}, \tag{6}$$

the state in this problem is the debt distribution $f_t\left(\cdot\right)$, making this problem intractable due to its high-dimensionality. Under perfect foresight, this problem can be avoided as calendar time t can be used to proxy for the dynamics of the distribution.

[6] Note that the government can hold foreign bonds ($f_t\left(\tau\right) < 0$) but in that case, foreign investors expropriate its assets if the country defaults.

is the price after the shock. The price after the shock (6) is the deterministic net-present value of the coupons and principal, using the discount rate process r_t^*. This reflects that after the shock, there is no default risk. Prior to the shock, the formula (5) includes two distinct elements. First, the Poisson intensity ϕ is added to the discount factor to discount cash-flows by the probability of the shock arrival. Second, there is an additional term reflecting the expected value of the bond upon the realization of the shock. It captures the discounted value of the post-shock bond multiplied by the probability that there is no default.

We can express equations (5) and (6) in PDE form as:

$$\hat{\xi}_t^*(\tau) \cdot \hat{\psi}_t(\tau) = \delta + \frac{\partial \hat{\psi}}{\partial t} - \frac{\partial \hat{\psi}}{\partial \tau}, \; ; \hat{\psi}_t(0) = 1, \tag{7}$$

and

$$r_t^* \cdot \psi_t(\tau) = \delta + \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial \tau}, \; ; \psi(0) = 1, \tag{8}$$

respectively.[7] On both PDEs, the first term in the right-hand side is the coupon flow, the second term is the expected capital gain over time, and the last term the change in value with maturity. The difference between both PDEs is just the discount rate employed. After the shock there is no additional risk, and hence investors employ the risk-free rate r_t^*. Before the shock, they employ the risk-adjusted rate:

$$\hat{\xi}_t^*(\tau) \equiv \underbrace{\hat{r}_t^*}_{\text{short-term rate}} + \phi \cdot \left(1 - \mathbb{E}_t^X \left[\underbrace{\Theta_t}_{\text{credit risk}} \cdot \underbrace{J^X[\psi_t(\tau)]}_{\text{interest risk}}\right]\right). \tag{9}$$

The risk-adjusted rate $\hat{\xi}_t^*(\tau)$ is the short-term rate plus a τ-specific risk premium. To understand the risk premium, suppose that default is certain upon the shock, $\Theta_t = 0$. In that case, the bond would be discounted by $\hat{r}_t^* + \phi$, capturing the probability of default, i.e. *credit risk*. Conversely, if repayment is certain, $\Theta_t = 1$, the risk premium is

$$\phi\left(1 - \mathbb{E}_t^X\left[J^X[\psi_t(\tau)]\right]\right) = \phi \mathbb{E}_t^X\left[\left(\hat{\psi}_t(\tau) - \psi_t(\tau, X)\right)/\hat{\psi}_t(\tau)\right],$$

[7] Appendix A includes the equivalence between PDE and integral representations with risk and default.

which captures the *capital loss* from the change in the price of τ-bond given a jump in the short-term rate r_t^*. This captures *interest-rate risk* which, naturally, the depends on τ because bonds of different maturity are impacted differently. In equilibrium, Θ_t is a function of the government's value, $V\left[\hat{f}\left(\cdot,t\right)\right]$. Thus, the general formula for $\hat{\xi}_t^*\left(\tau\right)$ couples together credit and interest rate risk: for each possible draw of X following a jump event, the multiplication of Θ_t times $J^X\left[\psi_t\left(\tau\right)\right]$ reflects the capital gains conditional on survival.

2.4 Government problem

The government's problem can be represented recursively:

Problem 1 (Government Problem)**.** *The government's solves the following problem*

$$\hat{V}_0 \equiv \hat{V}\left[f_0\left(\cdot\right),X_0\right] = \max_{\left\{\hat{\iota}_t(\cdot),\hat{f}_t(\cdot),\hat{c}_t\right\}_{t\geq 0,\tau\in[0,T]}} \mathbb{E}_0\left[\int_0^{t^o} e^{-\rho t}U\left(\hat{c}_t\right)dt + e^{-\rho t^o}\mathbb{E}_{t^o}^X\left[\max\left\{V_{t^o},V_{t^o}^D\right\}\right]\right] \tag{10}$$

subject to the law of motion of debt, (1), the budget constraint, (2), and the pricing equation (7). The post-default valueV_{t^o} is defined as

$$V_{t^o} = V\left[f_{t^o}\left(\cdot\right),X_{t^o}\right] = \max_{\left\{\iota_t(\cdot),f_t(\cdot),c_t\right\}_{t\geq t^o,\tau\in[0,T]}} \int_{t^o}^{\infty} e^{-\rho(t-t^o)}U\left(c_t\right)dt. \tag{11}$$

In this problem, the government cannot commit to repay its debt once the shock arrives or to a pre-specified path of debt after the shock. It can, however, influence these decisions by "future governments" by affecting the debt profile. As discussed by Davila and Schaab (2022), this is the equivalent of optimal policies under discretion.

3. Theoretical results

We now characterize the solutions to the optimal debt maturity and analyze the effects of different forces. All proofs are found in the appendix.

3.1 Optimal debt management

The following proposition gives the solution to the government problem.

Proposition 1. *(Necessary conditions) If a solution to the government's problem (10) exists, it satisfies the optimal issuance rule*

$$\hat{\iota}_t(\tau) = \underbrace{\frac{\hat{\psi}_t(\tau) - \hat{v}_t(\tau)}{\hat{\psi}_t(\tau)}}_{\text{pre-shock value gap}}, \quad \iota_t(\tau) = \frac{1}{\bar{\lambda}} . \underbrace{\frac{\psi_t(\tau) - v_t(\tau)}{\psi_t(\tau)}}_{\text{post-shock value gap}}, \tag{12}$$

where pre-shock domestic valuations $\hat{v}_t(\tau)$ satisfy the PDE

$$\hat{\xi}_t(\tau)\,\hat{v}_t(\tau) \;=\; \delta + \frac{\partial \hat{v}}{\partial t} - \frac{\partial \hat{v}}{\partial \tau};\ \hat{v}_t(0) = 1, \tag{13}$$

where the domestic discount $\hat{\xi}_t(\tau)$ is a risk-adjusted discount rate:

$$\hat{\xi}_t(\tau) = \underbrace{\rho + \frac{U''(c_t)\,c_t}{U'(c_t)}}_{\hat{r}_t} + \phi \left(1 - \mathbb{E}_t^X \left[\left(\underbrace{\Theta_t}_{\text{credit risk}} + \underbrace{\Omega_t}_{\text{revenue echo}} \right) \times \underbrace{J^X[v_t(\tau)]}_{\text{hedging}} \times \underbrace{J^X[U_t'(c_t)]}_{\text{self-insurance}} \right] \right),$$

The term Ω_t is the revenue echo:

$$\Omega_t \equiv \theta_t U'(\hat{c}_t) \times \int_0^T \int_{\max\{t+m-T,0\}}^{t} e^{-\int_z^t (r_u^* - \hat{r}_u)du} \times \frac{\psi_t(m)}{2\bar{\lambda}} \times \left(1 - \left(\frac{\hat{v}_z(m+t-z)}{\hat{\psi}_z(m+t-z)} \right)^2 \right) dz dm.$$

Post-shock domestic valuations $v_t(\tau)$ satisfy the PDE

$$\underbrace{\left(\rho - \frac{U''(c_t)\,c_t}{U'(c_t)} \frac{\dot{c}_t}{c_t} \right)}_{r_t} v_t(\tau) \;=\; \delta + \frac{\partial \hat{v}}{\partial t} - \frac{\partial \hat{v}}{\partial \tau};\ \hat{v}_t(0) = 1. \tag{14}$$

Proposition 1 shows that the government's problem can be decentralized, an extension allowing for risks to the deterministic case in Bigio et al. (2022). Under this decentralization, the optimal issuance rule displayed in equation (12), both for the pre- and the post-shock spells, shows that the government should issue debt proportionally to a *value gap*, defined as the difference between the market price of a bond of this

maturity, and a corresponding domestic valuation of that debt.

Post-shock valuations—given by eq. 14—should be priced using the domestic discount factor $r_t = \rho - \frac{U''(c_t)c_t}{U'(c_t)}\frac{\dot{c}_t}{c_t}$, which includes the subjective discount factor, the intertemporal elasticity of substitution, and the consumption growth rate. Valuations thus differ from (post-shock) market prices (given by eq. 8), which instead are priced using the risk-free rate r_t^*.

Decoupling Forces. The PDE for pre-shock valuations (13) and pre-shock prices (5) can be expressed compactly as

$$\hat{v}_t(\tau) = e^{-\int_t^{t+\tau}(\hat{r_s}+\phi)ds} + \int_t^{t+\tau} e^{-\int_t^s(\hat{r_u}+\phi)du}\left(\delta + \phi\mathbb{E}_s^X\left[\left(\Omega_{t+s}+\Theta_{t+s}\right)J^X\left[U_t'(c_t)\right]v_{t+s}(\tau-s)\right]\right), \tag{15}$$

and

$$\hat{\psi}_t(\tau) = e^{-\int_t^{t+\tau}(\hat{r_s}^*+\phi)ds} + \int_t^{t+\tau} e^{-\int_t^s(\hat{r_u}+\phi)du}\left(\delta + \phi\mathbb{E}_s^X\left[\Theta_{t+s}\psi_{t+s}(\tau-s,)\right]\right)ds. \tag{16}$$

This representation for valuations and prices capture the different forces that shape the optimal debt profile, (i) hedging, (ii) self-insurance, (iii) insurance through default (credit risk), and (iv) incentives to dilute debt (revenue-echo effect). We present a decomposition of each force next. We use this decomposition to make quantitative statements in the following section.

Hedging. Hedging refers to the desire to choose a debt profile to insulate from risks by generating "capital gains" in certain states. The desire to *hedge* risk is captures by the jump in valuations, $J^X\left[v_t(\tau)\right]$ in (13), vis-*à*-vis the jump in prices. In general, hedging is motivated by the desire against insurance motive, but the presence of liquidity frictions, by provoking differences in discount rates induce a desire to hedge refinancing needs.

To clarify why the jump in valuations captures hedging, we describe the solution to the model without liquidity costs and no default first, corresponding to the problems studied in Angeletos (2002) and Buera and Nicolini (2004).

The necessary conditions in Proposition 1 hold regardless of liquidity costs. How-

ever, without liquidity costs, if the issuance rule holds, issuances are bounded if and only if valuations and prices are equal, $v = \psi$ and $\hat{v} = \hat{\psi}$ as we take the limit $\bar{\lambda} \to 0$. Substituting $v = \psi$ and $\hat{v} = \hat{\psi}$, we obtain that the domestic discount factor and the international rate should coincide at all t:[8]

$$\hat{r_t} - \phi \mathbb{E}_t^X \left[J^X \left[U_t' \left(c_t \right) \right] \cdot J^X \left[\psi_t \left(\tau \right) \right] \right] = \hat{r_t}^* - \phi \mathbb{E}_t^X \left[J^X \left[\psi_t \left(\tau \right) \right] \right]. \tag{17}$$

Equation (17) can only hold if consumption does not jump upon the shock, $\hat{c} = c_{t^o} \left(X_{t^o} \right)$. Moreover, the equation implies that before and after the jump, consumption paths satisfy the usual growth rate under CRRA $\frac{\dot{c}_t}{c_t} = \frac{r_t^* - \rho}{\sigma}$. Intuitively, because international investors are risk-neutral and the government is risk-averse, the government's portfolio must replicate a complete markets allocation where risk-averse parties get full insurance. To construct a perfect hedge that fully insures the government, the debt profile must generate capital gains so that the government chooses the same consumption after the shock.

The capital gains needed to induce no jump in consumption can be computed as follows: first, we can compute the present value of consumption, solving first for the path of consumption after the shock. The initial condition for consumption is its value before the shock and the full path can be computed using: $\dot{c}_t = c_t \sigma^{-1} \left(r_t^* - \rho \right)$. Then, the capital gains for each possible shock, $\Delta B_{t^o} \left(X_{t^o} \right)$, must guarantee that each possible consumption path satisfies the government's intertemporal budget. A perfect hedge is possible if it is possible to solve for some $\hat{f}_t$ such that:

$$\Delta B_t \left(X_t \right) = - \int_0^T \left(1 - J^X \left[\psi_t \left(\tau \right) \right] \right) \psi_t \left(\tau ; X_t \right) \hat{f}_t \left(\tau \right) d\tau m, \quad \forall X_t \sim G(\cdot | \hat{X}_t). \tag{18}$$

The above equation is a functional equation in the sense that it holds for X_t. This functional equation is a generalization of the discrete-shock and discrete-bonds matrix conditions that guarantee market completion in Duffie and Huang (1985), Angeletos (2002) or Buera and Nicolini (2004).[9] Consider the case of a single jump event where

[8] This equation is recovered also by solving the problem with $\lambda = 0$ directly. The proof is available upon request.

[9] Angeletos (2002) or Buera and Nicolini (2004) obtain analogous conditions for the existence of a perfect hedge for the case where shocks and bonds are discrete. The existence of a perfect hedge requires the presence of at least $N + 1$ bonds and N distinct shocks and an invertibility condition for a matrix. In

interest rates jump and revenues fall: to offset the reduction in the net-present value of revenues, the debt profile must generate an increase in the government's portfolio. To produce capital gains, the government must hold a portfolio with short-term assets and long-term liabilities such that, upon the shock, reverting the positions produces capital gains.

With liquidity costs, a perfect hedge is impossible because there are costs to rebalance the debt profile. Nevertheless, the jump in valuations relative to the jumps in prices will tell us the direction in which portfolios should be rebalanced: consider what happens to issuances before and after a shock. If prices and valuations jump after a shock, the issuance rule tells the government to issue ex-ante the debt with the value gap that is expected to jump by more, i.e., akin to issuing more in the maturities that becomes "cheaper" for the government compared to the market, and less in maturities that become more expensive, a strategy akin to generate capital gains in adverse states.

Self-insurance. If the government could perfectly hedge against shocks with a bond portfolio, $J^X\left[U_t'\left(c_t\right)\right]$ in (13), would not jump. Because forming a perfect hedge may not be possible, the ratio of marginal utilities $J^X\left[U_t'\left(c_t\right)\right]$ in (13) jumps and captures the desire to induce *self-insurance*. This ratio is an "exchange rate" between states: the relative value of goods before *vis-à-vis* after the shock. For example, if a shock produces a drop in consumption, the ratio of marginal utilities in that state is greater than one.

To flesh out this force, let's consider again the case without liquidity costs, no default, and assuming constant interest rates so that $\psi = \hat{\psi} = 1$. This is a situation in which no hedging is possible, because it is impossible to generate capital gains when an income shock occurs. Instead, the only possible solution to (17) is:

$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = \frac{\hat{r}_t{}^* + \phi\left(\mathbb{E}_t^X\left[J^X\left[U_t'\left(c_t\right)\right]\right] - 1\right) - \rho}{\sigma}.$$

Self insurance is captured by the ratio of marginal utilities $\frac{U'(c_t)}{U'(\hat{c}_t)}$ which lowers $\hat{r}_t$.

the case of a continuum of shocks and maturities, Equation (18) is *Fredholm equation of the first kind*, the condition for existence of a solution requires the invertibility of the linear operator associated with this equation.

To solve for consumption, this extreme case coincides with a single-bond economy without interest-rate risk. The jump in consumption is given by the jump in the net present value of income. The solution to c_t, in this case, is known and can be found, for example, in Wang et al. (2016).

Credit risk. *Credit risk* is captured by the repayment probability, Θ_t, in (13). This term is also present in bond prices. In this environment, default allows some hedging. Consider the case without liquidity costs and income shocks. Without default, we noted that there was no hedging role for maturity, but now we show that with default, there is a role. Default opens the possibility of a partial hedging because, prior to the shock, different maturities are differently priced. Post-shock prices are always $\psi_{t^o}(\tau) = 1$. This means that, once a shock hits, the government can exploit the change in the yield curve to obtain capital gains in its portfolio. The change in the risk premium is akin to the spanning effect of an interest-rate jump.

Revenue echo. Finally, we encounter a term we dub the *revenue echo* Ω_t. The revenue echo captures how the government deals with the incentives to *dilute* the value of its debt before the default event. With endogenous default, when the government issues a τ-bond at time t, it factors in that the issuance of a specific bond will impact the probability of default until the bond expires. Thus, the increased likelihood of default will impact the auction revenues from time t to $t+\tau$. As the government can commit to a fiscal path before the shock arrival, it also takes into account the impact on the auction revenues of bonds issued prior to time t that have not yet expired. The revenue-echo effect Ω_t summarize these calculations into one term.

We develop a more detailed explanation with the aid of Figure 1. The x-axis represents time, and the y-axis represents maturity. Consider a marginal increase in the stock of τ_0-bonds at date t_0. The marginal issuance has a marginal increase in the profile $f_t(\tau)$ in every $\{t,\tau\}$ point along the $(1,-1)$ in the time-maturity direction, starting from $\{t_0,\tau_0\}$. Fix an arbitrary date in the future, denoted by $t > t_0$ in the figure. By time t, the bond has a maturity $\tau = \tau_0 - (t-t_0)$. Thus, the bond reflects the marginal impact on the repayment probability of a bond of maturity τ. If we multiply the term $\theta(V_t)U'(\hat{c}_t)$, inside Ω_t, by the term $\frac{U'(c_t)}{U'(\hat{c}_t)}v$ in the valuation, we obtain the marginal effect

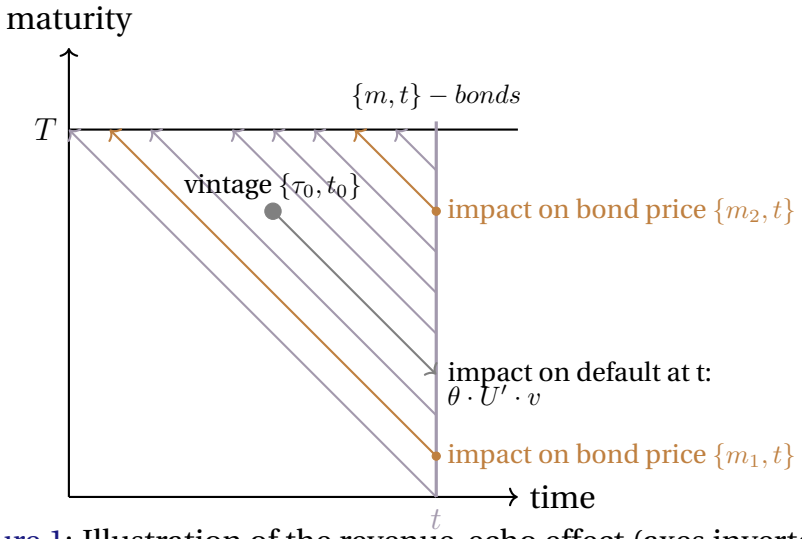


Figure 1: Illustration of the revenue-echo effect (axes inverted).

on the repayment probability of that bond:

$$\underbrace{\theta\left(V_t\right)U'\left(c_t\right)}_{\text{effect on repayment probability}}\cdot v_t\left(\tau\right).$$

Hence, the term $\theta\left(V_t\right)U'\left(c_t\right)v_t\left(\tau\right)$ reflects the marginal effect of the (τ_0,t_0)-bond, on the repayment probability by time t.

The double integral captures how a lower repayment probability at time t impacts the revenues generated by all issuances in all moments before t. Notice how the range of integration covers all maturities $m\in[0,T]$ in the outer integral. The inner integral covers the relevant times prior to time t, $z\in\max\{t+m-T,0\}$, when the marginal effect on repayment of the (t,τ)-bond affects the repayment probability of other bonds. These bonds are those that are still outstanding at time t, with a maturity m—represented graphically in the vertical line at time t. The effect of default at time t impacts past prices in the form of an "echo effect": Any bond price at a date prior to t for a bond that is still outstanding at time t should include the discounted value of its price at t, $e^{-\int_z^t(\hat{r}^*(u))du}\psi_t\left(m\right)$ for a specific maturity m. For example, the price of a bond with maturity m, $\psi_t\left(m\right)$, affects the price of all bonds of maturity $(m+t-z)$ at time z indexed by $z\in\max\{t+m-T,0\}$. Each ray that extends from the vertical line at t depicts one such family of bonds. Thus, if we multiply the change in the repayment probability at

t by $e^{-\int_z^t(\hat{r}^*(u))du}\psi_t(m)$, we obtain the reduction in the price of the $(m+t-z,z)$-bond. We can do the same for all bonds in $m \in [0,T]$, the outer integral, and past times z, the inner integral, to obtain the marginal effect that the (t,τ)-bond has on all past bond prices.

Naturally, the marginal impact on past prices affects past revenues. Thus, fix a maturity m and a date t. If we want to get the effect on revenues of a change in past prices, we must multiply the change in price by $\hat{\iota} - \left(\bar{\lambda}/2\right)(\hat{\iota})^2$, the issuance amount net of the liquidity costs. If we use the optimal issuance rule (12), revenues are proportional to:

$$\frac{1}{2\bar{\lambda}}\left[1-\left(\frac{\hat{v}_z(m+t-z)}{\hat{\psi}_z(m+t-z)}\right)^2\right].$$

Thus, when this term is inside the double integral, it captures the impact on past revenues of an increase in default probabilities. We bring past reductions in revenues into the current period t by multiplying by $e^{\int_z^t \hat{r}(u)du}$. The echo effect is present at any instant prior to the maturity of the bond, which is why it appears as a flow in equation (13). When the government considers the marginal issuance of the (τ_0,t_0)-bond, its valuation is the present value of all the echo effects Ω that lasts throughout the life of the bond. This reduction in revenues is part of the issuance consideration.

3.2 The risky steady state

Although Proposition 1 characterizes the solution with risk and default, which allows us to identify forces, calculating an exact solution is unfeasible. Because the state variable is the entire maturity distribution, the characterization of an equilibrium confronts the same computational challenges as heterogeneous-agent models with aggregate shocks: namely, the need to solve a dynamic programming problem for individual maturities which depends on the debt distribution at the time of the shock, a stochastic infinite-dimensional object. Although there are numerical approaches to deal with this issue, e.g., Krusell and Smith (1998) or Reiter (2009), here we are interested in an analytic characterization of the effect of various forces.

To this end, we introduce a risky steady state (RSS) approach that allows us to make

analytic progress. The RSS is the asymptotic limit before the jump realization. In other words, the RSS is the state of convergence of the solution, when the shock has been expected to arrive forever but has not yet materialized. Since the jump happens only once, the solution after the jump is the perfect foresight solution. This way, the distribution at the time of the shock arrival is stationary, and an exact computation to any desired level of precision is feasible. Furthermore, we can exploit the closed-form expressions to analytically characterize how different forces shape the maturity of debt.

The advantage of the approach is that we obtain an explicit solution to the valuations in the RSS. In the rest of the paper, we denote a variable x_t in the RSS by x_{rss}. Valuations (eq. 19) and prices (eq. 20) are

$$\hat{v}_{rss}(\tau) = \delta \cdot \int_0^\tau e^{-\hat{\xi}_{rss}\cdot(\tau-s)}ds + e^{-\hat{\xi}_{rss}\cdot\tau} \text{ and } \hat{\psi}_{rss}(\tau) = \delta \cdot \int_0^\tau e^{-\hat{\xi}^*_{rss}\cdot(\tau-s)}ds + e^{-\hat{\xi}^*_{rss}\cdot\tau},$$

where

$$\hat{\xi}_{rss}(\tau) = \rho + \phi\left(1 - \mathbb{E}^X\left[(\Theta_0 + \Omega_0) J^X\left[U_0' v_0(\tau)\right]\right]\right) \tag{19}$$

and

$$\hat{\xi}^*_{rss}(\tau) = \hat{r}^*_{ss} + \phi\left(1 - \mathbb{E}^X\left[\Theta_0 J^X\left[\psi_0(\tau)\right]\right]\right), \tag{20}$$

and $\Theta_0 = \Theta(V_0)$ and where $v_0(\tau)$ and $\psi_0(\tau)$ are the valuations of the *perfect-foresight problem* with initial conditions $\hat{f}_{rss}(\tau)$ and X_{rss}. The RSS can be computed as a fixed point in the space of consumption.[10]

We are interested in describing how the forces discussed above shape the aggregate volume of issuances and its weighted average maturity (WAM). The WAM is defined as

$$\text{WAM} \equiv \frac{\int_0^T \tau \iota_{rss}(\tau)\, d\tau}{\int_0^T \iota_{rss}(\tau)\, d\tau}.$$

In the absence of risk and default, the issuance distribution is given by

$$\iota_{rss}(\tau) = \frac{1}{\bar{\lambda}}\left[1 - \frac{e^{-\rho\tau} + (\delta/\rho)\left(1 - e^{-\rho\tau}\right)}{e^{-r^*_{rss}\tau} + (\delta/r^*_{rss})\left(1 - e^{-r^*_{rss}\tau}\right)}\right], \tag{21}$$

[10] The algorithm proceeds as follows: .

provided that liquidity costs are large enough.[11] In steady state, issuances are spread across maturities in an increasing way, as long as the government is more impatient than investors, $\rho > r^*_{rss}$. Notice how $\lim_{\tau\to 0} \iota_{rss}(\tau) = 0$ and $\lim_{\tau\to\infty} \iota_{rss}(\tau) = \frac{1}{\lambda}\left(1 - \frac{r^*_{rss}}{\rho}\right)$. Next, we describe how risk and default affect the issuance distribution. To this end, we consider small perturbations around the RSS to isolate the different forces.

Proposition 2. *[**Stochastic Forces**] Consider a RSS with $0 = \delta < r^* < \rho$, when liquidity costs are arbitrarily large, $\bar{\lambda} \to \infty$ (zero-liquidity limit). We have the following:*

1. *[Self-Insurance] Assume that default is not possible, $\Theta_0 = 1,\ \Omega_0 = 0$. Consider an expected small negative shock to income, $\varepsilon = y_{ss} - y_0$, which reverts back immediately to its steady state value, i.e., $\alpha^y \to \infty$. Then, issuances in the RSS decrease compared to the case without shocks. If $T < \tau^* \equiv \frac{1}{\phi}\log\left(\frac{\phi+(\rho-r^*_{ss})}{(\rho-r^*_{ss})}\right)$, then the WAM increases in the RSS.*

2. *[Credit Risk] Let the government be risk-neutral, $\sigma = 0$. Consider a small increase in the default probability, $\hat{\Theta}_0 = \Theta_0 + \varepsilon$. The issuance distribution does not change compared to the case without shocks.*

3. *[Revenue Echo] Let the government be risk-neutral, $\sigma = 0$. Consider small increase in the revenue echo, $\hat{\Omega}_0 = \Omega_0 + \varepsilon$. Then issuances in the RSS decrease compared to the case without shocks. If the default probability is very low, $\Theta_0 \approx 1$, then the WAM increases in the RSS.*

Proposition 2 describes how the different forces impact the RSS issuances and their WAM. We consider the large liquidity-cost limit $\bar{\lambda} \to \infty$ where issuances vanish.

The first item characterizes the effect *of self-insurance* in isolation because no term in equations (19) responds to an instantaneous income shock except for the jump in marginal utilities, $J^X[U'_0]$. This term enters the valuation equation (20) in the same way as the impatience coefficient ρ, but with the opposite sign. Thus, the effect of the shock is similar to reducing the domestic discount rate, that is, making the government more patient. A more patient government reduces the issuances and lengthens the WAM: a more impatient government is willing to accept lower prices at all maturities, but this

[11] The proof and the conditions for the existence of a deterministic steady state can be found in Bigio et al. (2022).

impact is especially strong at short maturities, as discussed by Bigio et al. (2022). [12] These are hallmarks of self-insurance behavior: taking less debt to reduce payments and extending the maturity to avoid large principal payments when income is low.

The second item describes the effect of *credit risk* in isolation. Because we set risk aversion to zero, $\sigma = 0$, and liquidity costs are arbitrarily high, $\bar{\lambda} \to \infty$, the revenue echo is negligible: an increase in the default probability is then akin to introducing a lottery that erases debts. In this case, risk enters both in the valuation (eq. 19) and price equations (eq. 20), and these terms cancel each other in the issuance rule. This is a natural outcome since both the government and investors are risk-neutral. Therefore, credit risk alone does not affect the issuance distribution under risk neutrality because it becomes akin to a fair bet.

Things are different in the risk-averse case, $\sigma > 0$. If the government is risk-averse, default is a form of insurance (an idea found, for example, in Zame, 1993); thus, default reduces the impact of self-insurance. We can see this in the valuations in (equation 19), because greater credit risk reduces the effect of the jump in marginal utilities. We can establish that credit risk, per se, has no effect on issuances. Coupled with risk aversion, credit-risk induces greater issuances at shorter maturities as it mitigates the need for self-insurance.

The final item characterizes how changes in the *revenue echo effect* impact issuances and maturity. An increase in the revenue echo by ε captures the indirect effect that any shock will have on the default probability, and how that impacts past and future auction revenues, as we explained above. The revenue echo term, Ω, is positive, and thus it makes the government more patient. As we showed earlier, as the government becomes more patient, it issues less at all maturities. Hence, there is a negative impact on issuance. We also showed that, as the government becomes more patient, the issuance of short-term bonds is more sensitive in relative terms. Thus, the revenue echo effect decreases issuances and increases the average maturity.

All in all, the effect of shocks can be decomposed into the effects of self-insurance, credit risk, the revenue-echo effect, and hedging. A virtue of our approach is that we carry out this decomposition quantitatively. Another advantage of our approach is that

[12] Notwithstanding, the effect of self-insurance on the WAM can be ambiguous if $T \geq \tau^*$, as the issuance elasticities can be negative for maturities close to expiration.

we can study the effects of small perturbations around the RSS. In the following sections, we build on the results so far and develop these quantitative applications.

4. Calibration

We calibrate the model to Spanish data from January 2002 to April 2018 using the dataset in Bigio et al. (2022), which covers all sovereign issuances over the period and reports primary- and secondary-market prices. Spain is a useful case study for two reasons. First, despite being an advanced economy, markets price a non-zero default probability—as reflected in Spanish yields relative to other euro-area bonds—so default considerations are empirically relevant for the Treasury. Second, monetary policy is set by the European Central Bank, allowing us to treat Spain's (real) interest rates as outside the government's control.

We modify the government's problem so that it issuances in the discrete maturities of the set $\mathcal{M}$, including bonds of 3, 5, 10, 15, and 30 years.[13] The government's budget constraint (2) is modified to:

$$c_t = y_t - f_t(0) + \sum_{\tau \in \mathcal{M}} \left[q_t(\tau, \iota) \iota_t(\tau) - \delta f_t(\tau) \right]. \tag{22}$$

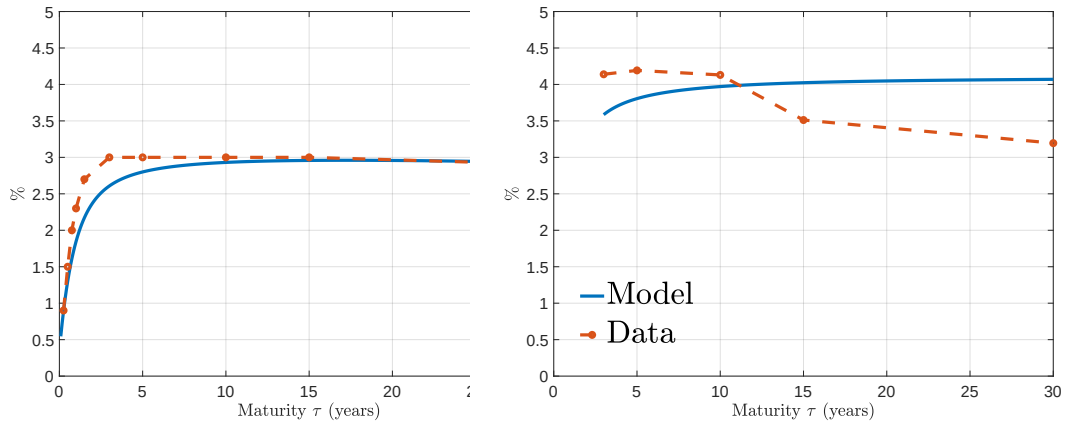
We solve the model using a finite-difference method, interpreting the frequency as monthly, $\Delta t = \Delta \tau = 1/12$, so that, consistent with the data, bonds are issued every period.

Liquidity costs. Consistent with our specification with liquidity costs, Spain regularly issues debt in the subset of maturities in $\mathcal{M}$. We modify the liquidity costs, $\bar{\lambda}(\tau)$, in proposition 1 so that they now vary with maturity, as in the data. The optimal issuance rule is thus modified by replacing $\bar{\lambda}$ by $\bar{\lambda}(\tau)$ of the corresponding maturity. The values are taken from the estimations in Bigio et al. (2022), $\bar{\lambda}\left(\tau \in \{36, 60, 120, 180, 360\}\right) = 2\Lambda\left(\tau\right) = 2 \cdot 10^{-2} \cdot \{8, 12, 17, 40, 56\}$.

[13] We discard both bills (maturities below 1.5 years), as there are liquidity-management considerations that affect the rationale behind their issuances (see) and 50-year bonds as they are sparsely issued in this period.

The rest of the parameters and their values are presented in Table 1.

Coupons. We set the (real) coupon rate δ to 2.4 percent, based on the average nominal coupon rate of 4.2 percent in the period, which we compute from the microdata, and the average 1-year-ahead inflation expectation of 1.8 percent, derived from inflation-linked swaps (ILS).



(a) Bonds (b) Domestic valuations

Figure 2: Zero-coupon yield curves of bonds and domestic valuations.

Notes: Panel a: Zero-coupon real yield curve for Spain in the RSS. The empirical counterpart is constructed as the difference between the nominal zero-coupon yield curve on Spanish debt minus the expected euro inflation curve, derived from ILS plus 32 bps to all maturities to compensate for a constant in the auction markup estimate in Bigio et al. (2022). Panel b: Zero-coupon domestic valuations yield curve for Spain in the RSS. The empirical counterpart is constructed using the optimal issuance formula with bond prices, issuances, and liquidity costs taken from the data.

Economic Disasters and interest rates shocks. Following Barro (2006), we consider the possibility of large crises in which revenues collapse or the short-term rate spikes. With these disasters, we capture events with meaningful declines in fiscal revenues, like those observed during the 2008 financial crisis or the COVID-19 crisis, and flight-to-quality episode. To this end, we consider two shocks: a negative income shock and a positive interest rate shock. We set the average income y_{ss} to one as a normalization. In the case of the income shock, we consider a 5 percent fall in revenues, corresponding to a large recession in the data in Barro and Ursua (2010). The average frequency is set

to 10 years, roughly a major shock per decade.[14] We set $\alpha^y \doteq 0.2$ to match estimates of the persistence.[15] The parameters of the rate shock are described below, as they are employed to match the yield curve.

Several parameters are calibrated to reproduce the yield curve. We construct daily estimates of the Spanish and French *nominal* yield curves and average them to obtain the average nominal yield curve in both countries.[16] We also construct an expected inflation curve using the ILS. We then create the Spanish and French zero-coupon *real* yield curves as the difference between the average nominal zero-coupon yield curve minus the average expected euro inflation curve. We interpret the French real curve as the *risk-free curve* and treat the difference as the default premium of Spain.[17] We display the real yield curve for Spain in the dashed red line in Panel (a) of Figure 2.

We calibrate the real short-term rate process to produce the RSS risk-free yield curve, which anticipates the disaster event. We set the steady-state real rate r^*_{ss} to -0.4 percent, such that the 1-year yield on risk-free bonds in the model is 1.6%, coinciding with the real 1-year yield on our target for the risk-free yield curve (not shown). The rest of the curve is generated with a limited number of parameters. We calibrate the arrival rate of the shock ϕ to 2 to replicate the slope of the yield curve at the short-end (0-18 months), and the persistence α^r to 0.0004, to replicate the almost-flat slope at the long-end (15-30 years). We set the after-shock value $r^*_{t_0}$ to 3.3 percent, such that the 30-year real rate is 2.9 percent (5.2 nominal rate - 2.3 expected inflation). This process generates a yield curve that well approximates the yield curve we construct from the data—the solid blue line in Panel (a) in Figure 2 corresponds to the yield curve generated by this process.

Credit risk. Regarding the parameters that govern credit risk, the output loss after default, κ, is set to 2 percent, following Aguiar and Gopinath (2006). Following the discrete-choice literature, the distribution of the shift to the autarky value, ε, is a logistic

[14] This is used to calibrate the probability of an income shock conditional on the arrival of a shock.

[15] The detrending uses a Hodrick-Prescott filter with a parameter of 1600. We estimate an AR(1) process $\log y_t = \rho_y \log y_{t-1} + \sigma_y \epsilon^y_{t-1}$ for the detrended seasonally adjusted output. The estimated persistence of quarterly income (ρ_y) is 0.95. This corresponds to a value of $\alpha^y = (1-\rho_y)/(3 \times \Delta t) = 0.2$.

[16] We employ a dynamic 3-factor model provided to us by Jens Christensen.

[17] The German yield curve is often considered the risk-free curve in the Euro area. Notwithstanding, the combination of fiscal surpluses and asset purchases by the ECB in some periods has produced some scarcity of German public debt in the global market, which may distort bond prices. This is why we consider the French debt, which typically has traded with a small constant spread over German debt.

with coefficient ς. We set ς to 2.3, to reproduce a 3-month bond credit risk premium of 60 bps. The unconditional default probability is 0.56 percent—roughly one default every 190 years, close to the reported frequencies in Reinhart and Rogoff (2009).

Preferences. We assume CRRA preferences, $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$. We set the inverse of the intertemporal elasticity of substitution, σ, to 2, a standard value. Finally, we calibrate the subjective discount factor, ρ, to replicate the average of total annual issuances over GDP (17 percent).[18] We obtain the best fit when we set ρ to 4.1 percent.

Valuations. Panel (b) of Figure 2 compares the domestic valuations in the model (solid blue line) with their empirical counterparts (dashed red line), constructed by inverting the optimal issuance rule, $v(\tau) = \psi(\tau) - \bar{\lambda}(\tau)\iota(\tau)$, where $\psi(\tau)$ is the market price of a zero-coupon bond in the data, $\bar{\lambda}(\tau)$ are the empirical liquidity costs and $\iota(\tau)$ the average issuances per maturity over the period. There, it can be seen that the model predicts lower valuations than the data for maturities below 10 years, and higher valuations for maturities above 10 years. This will lead to lower issuances for short-term bonds and higher for long-term bonds, as we discussed in the next section.

Finally, we note that systematic government bond buybacks are rare in Spain (also a feature in the US, Faraglia et al., 2018). This pattern is consistent with the current calibration as $\iota > 0$ for all maturities in every exercise. Hence, reductions in the WAM follow from changes in the maturity of issuances, but not from bond repurchases.

[18] To make the mapping of the model to the data more realistic, we replicate exogenously the issuance of bills, that is, bonds with maturity below 1.5 year so that we do not need to subtract this amount from the total issuance volume. This is inconsequential for the results.

	Description	Value	Target / source
Bonds			
T	Maximum maturity (years)	30	Maximum maturity of Spanish bonds
r^*_{ss}	Steady-state real rate (p.a)	-0.4%	1-year risk-free real rate of 1.6%
ϕ	Shock arrival rate	2	Slope of the yield curve (0-18 months)
$r^*_{t_0}$	After-shock rate (p.a)	3.3%	30-year real rate 2.9%
α^r	Rate shock persistence	0.0004	Slope of the yield curve (15-30 years)
δ	Coupon (p.a)	2.4%	Average real coupon 2.4%
Income			
y_{ss}	Steady state income	1	
y_{t_0}	After-shock income	0.95	Large recession in Barro and Ursua (2010)
α^y	Income shock persistence	0.2	GDP shocks persistence
Default			
κ	Post-default output loss	2%	Literature
ς	Parameter preference shock	2.3	3-month default premium 60 bp
Preferences			
σ	1/IES	2	Literature
ρ	Discount rate (p.a.)	4.1%	Annual issuance over GDP 17 percent

Table 1: List of calibrated parameter values.

5. Quantitative Applications

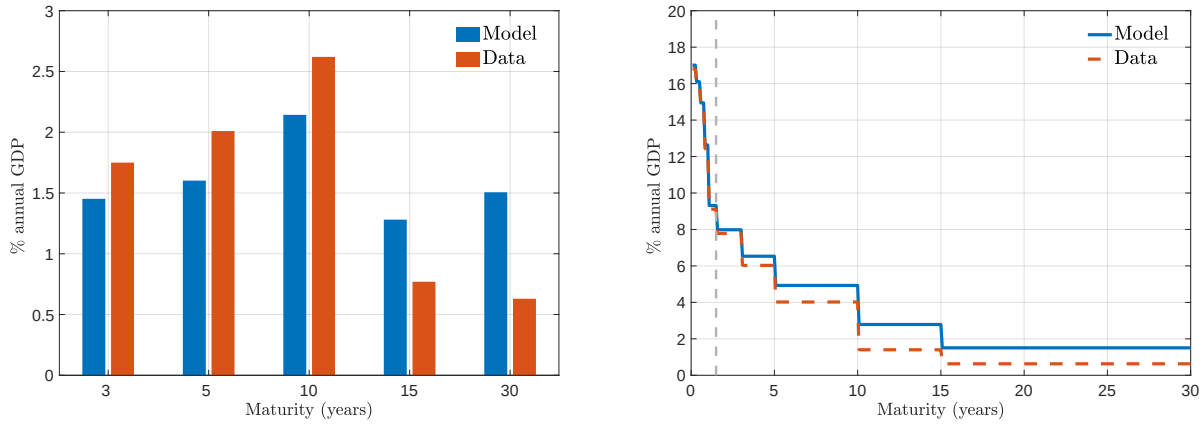
5.1 Evaluation

Armed with the calibration, we parse out whether, given the parameters that rationalize the yield curve and Spain's average debt to GDP ratio during the period, the average issuance and debt profiles in Spain are close to optimal vis-à-vis the profile generated by the model. Recall that we have deduced only one preference parameter, ρ, to match total issuances. However, the issuance profile contains information on average issuances for five exogenous maturities (3, 5, 10, 15, and 30 years), leaving us with four untargeted moments.

Panel (a) in Figure 3 contrasts the average annual *issuances* over GDP by maturity in the data against the corresponding values in the RSS of the model. Panel (b) compares the average *outstanding* debt profile in the data and the model.[19] The model replicates

[19]For consistency, we construct the data counterpart as if the observed average issuances are maintained indefinitely.

Spain's debt-management strategy, though it is slightly biased towards long maturities. In particular, the model reproduces the hump-shape issuances that peak at ten-year maturities. The WAM of issuances produced by the model, 6.2 years, is higher than the one observed in the data, 4.7 years. As a result, the model has a distribution of *outstanding* debt with more debt at the curve's long end.



(a) Issuances (b) Debt stock

Figure 3: Issuance and debt stock by maturities.

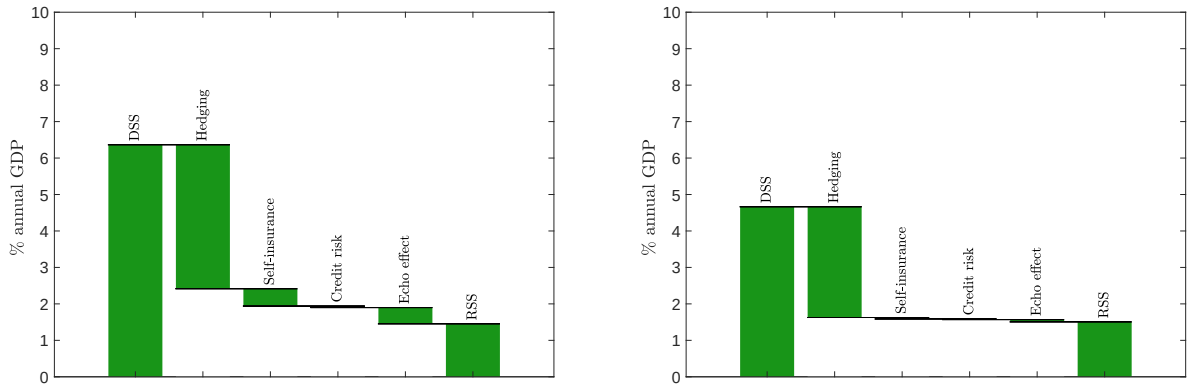
Notes: Panel a: Average yearly issuances as a percentage of annual GDP (data) versus RSS issuances (model). Panel b: Long-run debt stock, computed using perpetual inventory from issuances. Recall that bill issuances are fitted by construction as they are exogenously given.

5.2 Force decomposition

Next, we present a force decomposition to shed light on why the model makes this recommendation. The force decomposition builds on Proposition 1, and consists of progressively turning the terms associated with each force. To do this, we compute counterfactual values for domestic valuations v and market prices ψ by suppressing different terms in the formulas in proposition 1, and use the optimal issuance rule to calculate counterfactual issuances that isolate particular forces.

We present the decomposition outcomes through the waterfall chart in Figure 4: For ease of exposition, we perform the decomposition for only three maturities, namely the 3-, 10-, and 30-year issuances. 3-year and 30-year bonds are displayed in panels (a) and

(b), respectively, whereas 10-year bonds are shown in Appendix **??**. The decomposition corresponding to 5- and 15-year bonds is close to those for 3 and 30-year bonds, respectively. The first column ('DSS') represents the deterministic steady-state issuances, which are entirely pin down by the relative impatience and liquidity costs: we compute the issuances that we we would obtain if $\hat{\xi}_{rss}(\tau) = \rho$ and $\hat{\xi}^*_{rss}(\tau) = r^*_{ss}$, as in the deterministic steady-state. Each subsequent bar to the right of the panels represents the change in the issuance produced by a force. We describe the effects subsequently.



(a) 3-year bond (b) 30-year bond

Figure 4: Force decomposition.

Notes: Panel a: Decomposition of the RSS issuance of 3-year bonds. Panel b: Decomposition of the RSS issuance of 30-year bonds.

Hedging. The second columns in the panels of Figure 4 account for the impact on issuances due to hedging. As discussed above, perfect hedging is not possible with liquidity costs. However, the government may set its debt profile taking into account expected changes in market prices and valuations due to shocks. In this case, we compute the domestic discount using, $v_{hedging}$, using $\hat{\xi}_{rss}(\tau) = \rho + \phi\left(1 - \mathbb{E}^X\left[J^X\left[v_0(\tau)\right]\right]\right)$, that is, including the adjustments due to risk, but excluding the default, echo effect or self-insurance terms. Similarly, we compute the market price, $\psi_{hedging}$, using $\hat{\xi}^*_{rss}(\tau) = r^*_{ss} - \phi\left(1 - \mathbb{E}^X\left[J^X\left[\psi_0(\tau)\right]\right]\right)$. Figure 5 displays both objects as a ratio of the deterministic steady state counterparts. This way, values of the ratio $\frac{\hat{v}_{hedging}(\tau)}{\hat{v}_{dss}(\tau)}$ above one reflect maturities with higher valuations, and the same happens for $\frac{\hat{\psi}_{hedging}(\tau)}{\hat{\psi}_{dss}(\tau)}$.

Hedging reduces the desire to issue bonds at all maturities, but relatively more for the long maturities. As a result, the WAM falls from 18.87 years to 6.34 years. To understand the result, assume a zero-coupon bonds. In this case, optimal RSS issuances are

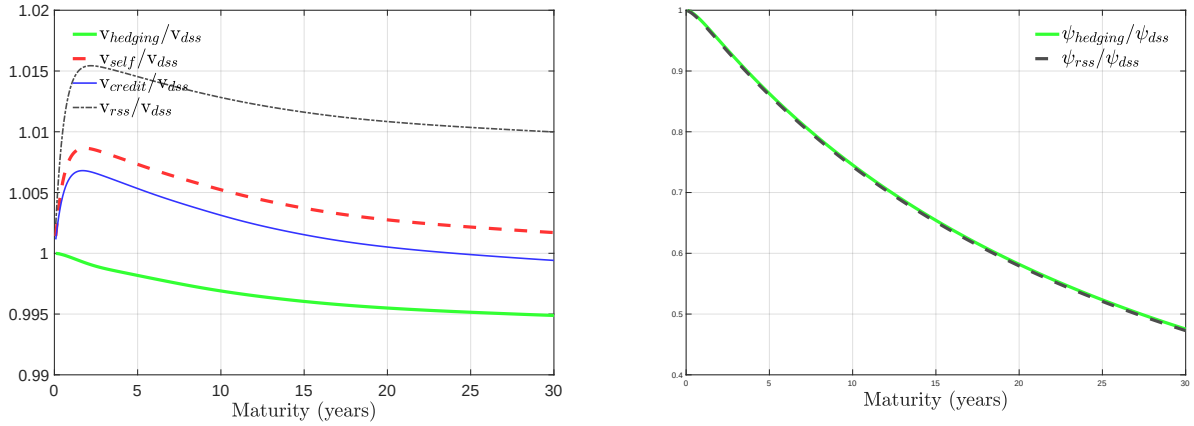
$$\begin{aligned}\iota_{rss}(\tau) &= \frac{1}{\bar{\lambda}(\tau)}\left[1-\exp\left(\int_0^\tau\left(\hat{\xi}^*_{rss}(s)-\hat{\xi}_{rss}(s)\right)ds\right)\right]\approx\frac{1}{\bar{\lambda}(\tau)}\int_0^\tau\left(\hat{\xi}_{rss}(s)-\hat{\xi}^*_{rss}(s)\right)ds\\ &= \frac{\rho-\hat{r}^*_{ss}}{\bar{\lambda}(\tau)}\tau+\frac{\phi}{\bar{\lambda}(\tau)}\int_0^\tau\left(\mathbb{E}^X\left[J^X\left[\psi_0(s)\right]-J^X\left[v_0(s)\right]\right]\right)ds\\ &\approx \iota_{ss}(\tau)+\frac{\phi}{\bar{\lambda}(\tau)}\int_0^\tau\left(\mathbb{E}^X\left[J^X\left[\psi_0(s)\right]-J^X\left[v_0(s)\right]\right]\right)ds.\end{aligned}$$

In our calibration, the expected jump in prices $J^X\left[\psi_0(s)\right]$ is large: after the shock, interest rates may persistently increase from -0.4% to 3.3%. This interest rate risk disproportionately affects long-maturity bonds than short ones. The reason is that the longer the maturity, the more likely it is that the shock arrives before the bond expires. This can be seen in Panel (b) of Figure 5: bond prices are much lower the longer the maturity compared to the flat yield curve in the deterministic steady state. Bear in mind that the decline in bond prices is the counterpart of the upward-sloping yield curve in Figure 2: the shape of the Spanish yield curve mimics closely that of the risk-free euro area yield curve.

The expected jump in valuations $J^X\left[v_0(s)\right]$ is much smaller than that in prices. The reason is that, being risk averse, the government, the domestic discount factor, $r_t = \rho - \frac{U''(c_t)c_t}{U'(c_t)}\frac{\dot{c}_t}{c_t}$, jumps by less than the rates after the shock. This can be seen in the ratio of valuations $\frac{\hat{v}_{hedging}(\tau)}{\hat{v}_{dss}(\tau)}$ in Panel (a) of Figure 5: valuations are much lower the longer the maturity, for the same reason as prices, but quantitatively, the decline is less than 0.5%, compared to more than 50% for prices. As the expected fall in prices is much larger than that of valuations, $\mathbb{E}^X\left[J^X\left[\psi_0(s)\right]\right] < \mathbb{E}^X\left[J^X\left[v_0(s)\right]\right] < 0$, the contribution of interest rate risk is negative. This explains the decline in issuances, especially in the long maturities.

The intuition for the short-term tilting in the maturity is the following. The government is trying to reconcile two objectives. On the one hand, the government is relatively impatient compared to the market wants to issue relatively more at long maturities in the presence of liquidity costs. This mechanism is already present in the

deterministic steady state. On the other hand, the upward-sloping yield curve due to interest-rate risk makes long-term bond prices less attractive to issue. Of course short-term bonds are subject to rollover risk if the shock arrives their price will collapse. The result is that the government issues less debt, given the adverse pricing, but this effect is more marked at long-term bonds.



(a) Valuations (b) Prices

Figure 5: Decomposition .

Notes: Panel a: Domestic valuations for different cases. Panel b: Market prices for different cases.

Self-insurance. The third column in Figure 3 captures the effect of self-insurance. In this case, bond prices are the same as above, $\hat{\psi}_{hedging}(\tau)$, but valuations $\hat{v}_{self}(\tau)$ are now based on $\hat{\xi}_{rss}(\tau) = \rho + \phi\left(1 - \mathbb{E}^X\left[J^X\left[U_0' v_0(\tau)\right]\right]\right)$, including the jump in marginal consumption utilities. In Proposition 2 we showed how, in the case of a negative income shock, self-insurance prescribes a reduction in issuances and an increase in the WAM. Here, agents expect both an income and an interest rate shock, so we have no theoretical guide. Our results indicate that the government reduces issuance volumes and marginally reduces the WAM, from 6.34 to 6.33 years.

As discussed above, bond prices coincide with those in the 'hedging' case. All the action comes from valuations. Panel (a) from Figure 5 shows how valuations increase at all maturities, which explains why issuances decrease. Though the increase in valuations is higher for short maturities, explaining the higher effect for 3-year bonds in

Figure 3, this is not enough to avoid the slight decrease in the WAM.

Credit-risk. The fourth column incorporates credit risk, but not the revenue echo effect, by including the repayment probability Θ_{rss} in both market and domestic discounts,

$$\hat{\xi}^{*}_{rss}(\tau) = r^{*}_{ss} - \phi\left(1 - \mathbb{E}^{X}\left[\Theta_{rss} J^{X}\left[\psi_{0}(\tau)\right]\right]\right),$$

and

$$\hat{\xi}_{rss}(\tau) = \rho + \phi\left(1 - \mathbb{E}^{X}\left[\Theta_{rss} J^{X}\left[v_{0}(\tau)\right]\right]\right).$$

In this case, both valuations $\hat{v}_{credit}(\tau)$ and prices $\hat{\psi}_{rss}(\tau)$ decrease, as shown in Figure 5. Their decrease has opposite effects on issuances, which almost cancel each other, in line with the results in Proposition 2. The impact on prices is lightly higher, which explains why issuances fall a bit at all maturities. The WAM also falls marginally from 6.33 to 6.30 years.

Revenue echo. Finally, the fifth column in Figure 3 takes into account the revenue echo-effect, Ω_{rss}, in valuations, $\hat{v}_{rss}(\tau)$. It operates in a similar way to how self-insurance does, as shown in Figure 5. It increases valuations, which in turn reduces issuances, especially for short maturities. This again is not enough to avoid a small reduction in the WAM to 6.23 years.

The decomposition above quantifies the impact of the different forces. The main conclusion from the exercise is that the most sizable force in shaping the distribution of debt maturity is hedging: the less relevant one is credit risk. Notwithstanding, it is important to highlight that these results depend on our particular calibration. One aspect particularly relevant for these results is that the slope of the yield curve has been generated through the risk of a highly persistent rise in interest rates, which helps to explain why hedging is so relevant. Under the alternative assumption that risk-averse arbitrageurs produce the term premium, as is the case, for instance, in Vayanos and Vila (2021), this result might be altered.

5.3 Application: Higher long-term rates and a 50-year bond.

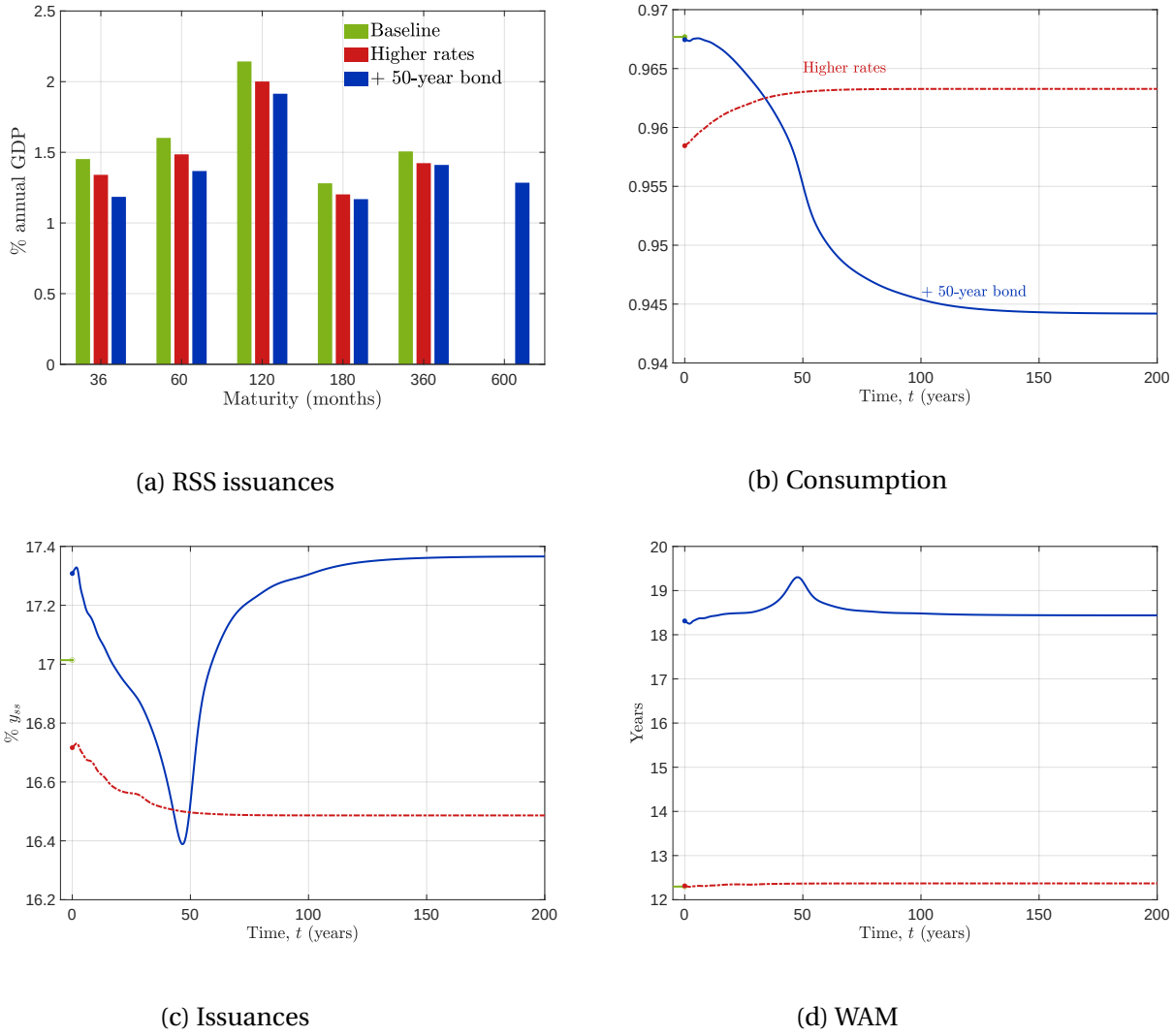
The fiscal stress provoked by the COVID-19 crisis and, more importantly, by the normalization of monetary policy across the World has brought new considerations to debates on optimal debt-management practices. Two themes stand out: first, the possibility of the transition from the low-interest environment that has characterized the last decade to a higher-interest scenario, and second, the possibility of increasing the set of available debt instruments with the introduction of bonds of very long maturity, such as fifty-year or hundred-year debt. We now exploit the framework to compute optimal debt strategies in these novel environments. The analysis of the introduction of a novel bond can only be conducted using a framework like ours.

Through the model, we can interpret the transition to a high-interest-rate environment as a permanent increase in both the steady-state short-term rate, r^*_{ss} and ρ. Thus, we modify the previous approach to study the transition from an initial RSS to a terminal RSS.

To compute a transition, we apply a similar concept to that of the "sequence space" solution of heterogeneous-agent models pioneered by Boppart et al. (2018) and Auclert et al. (2019). The idea of those papers is to analyze transitional dynamics around the deterministic steady state. Here, instead, we analyze transitional dynamics around the risky steady state. To this end, we first compute both the initial and the terminal RSS. Then, starting from the debt profile of the initial RSS, we compute the path of consumption consistent with issuance rules given by the price and valuation equations (7) and (13) where the jump terms accounting for the forces (hedging, self-insurance, credit risk, and the echo effect) are kept constant to their values in the terminal RSS.

Using this approach, we study a transition from the original RSS presented above to an RSS with lower rates. In particular, we study a 1pp increase in long-term rates—engineered via an increase in the jump in short-term rates from 3.3% to 4.5—together with a 1 pp rise in ρ. We compare the transitional dynamics to a high-interest-rate environment both in the baseline and in a scenario where Spain can also issue 50-year debt.[20]

[20] To study the effects of introducing 50-year bonds, we modify $\bar{\lambda}(\tau)$. In particular, we use the parametric estimation of $\bar{\lambda}(\tau)$ to obtain a value for $\bar{\lambda}(600)$. According to our microfoundation, the value of $\bar{\lambda}(600)$ is associated with a customer flow μ. Thus, we correct all $\lambda(\tau)$ to keep a constant total customer



(a) RSS issuances

(b) Consumption

(c) Issuances

(d) WAM

Figure 6: RSS and transitional Dynamics to low interest-rate environment (dashed red) and additional long-term bond (solid blue). The initial state is plotted in green.

The results are presented in Figure 6. Panel (a) shows the issuance profile for the high-interest-rate environment with and without the 50-year debt. As we can observe, the high-interest environment induces lower issuances at all maturities: the issuance flow increases from 17.0% to 16.5% percent of GDP, and the WAM increases marginally from 12.3 to 12.4 years.[21] More striking differences emerge when we introduce 50-year

flow. Because the linear form of $\bar{\lambda}$ is steep, the customer flow associated with a 50-year debt is miniscule and has only a minor impact on other liquidity coefficients.

[21] In this exercise, we assume that the government does not issue any bill, which explains why the debt and WAM numbers differ from those in the previous section.

debt. First, although the liquidity costs associated with 50-year debt are almost 50 percent higher than that of 30-year debt, the greater relative impatience of the government also produces a higher value gap, which leads to issuances in 50-year debt of a similar amount to 30-year debt. As a result, issuances over GDP increase to 17.4% and the WAM to 18.4 years.

Panels (b), (c), and (d) illustrate the path of consumption and the level and WAM of issuances, respectively, throughout both transitions to the new regimes. Let us first discuss the transition to a high environment—depicted in the red dashed line. The shock raises rates but also makes Spain more impatient. The shock provokes a lower issuance level and a reduction in the WAM. Interestingly, on impact, the shock induces a consumption bust. This is a consequence of the higher cost of debt funding, which forces the economy to reduce its debt rollover by increasing savings. After the initial bust, consumption increases toward a higher expenditure level, though still lower than in the baseline, as discussed above. The WAM remains practically unaffected.

Consider now the transition that occurs in tandem with the possibility of issuing fifty-year debt. In this case, the initial bust is negligible, but the economy returns to a much lower consumption level. As we noted, opening up markets for 50-year debt induces the country to substantially increase its issuances. Interestingly, the initial boom in debt issuances leads to a substantial increase in the debt principal 50 years later. After those fifty years, Spain will have issued continuously in fifty-year debt, so the debt service sequence will be smoother after that. The path of consumption initially decreases gently, but eventually reaches a lower point as the 50-year debt begins to accumulate, which forces the country to reduce its consumption more aggressively to pay for the increase in debt service. The transition toward lower consumption is prolonged, taking almost a hundred years until the economy reaches its new RSS consumption level.

6. Conclusions

Approach. We introduce a framework to study optimal debt-maturity management with liquidity costs in the presence of anticipated risk and default. The core difficulty is

that the state variable—the maturity distribution of outstanding debt—is an infinite-dimensional object. We make progress by exploiting a *risky steady state* (RSS): the asymptotic pre-shock limit of an economy that anticipates a large absorbing shock. Around the RSS we obtain closed-form pricing and valuation objects, a value-gap characterization of optimal issuance, and tractable transitional dynamics. Beyond the application here, the RSS device is a general tool for models with infinite-dimensional states and aggregate risk (e.g., heterogeneous-agent environments).

Scope and extensions. As a first step, our framework has limitations that suggest natural extensions. (i) We obtain exact solutions under a single absorbing shock, with default only at the shock; allowing recurrent shocks along the lines of Valaitis and Villa (2022) would enable richer dilution and default dynamics and relax the commitment-to-issuance assumption prior to the shock. (ii) We adopt a specific, issuance-dependent liquidity-cost specification; extending to stock- and maturity-dependent liquidity as in Vayanos and Vila (2021) is feasible within our approach. We leave these extensions for future work.

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Appendix

Contents

A. Equivalence between PDE and integral formulations

Valuations and prices are given by continuous-time net present value formulae. Their PDE representation is the analogue of the recursive representation in discrete time, and the integral formulation is the equivalent of the sequence summations. The solutions to each PDE can be recovered easily via the method of characteristics or as an immediate application of the Feynman-Kac formula. All of the PDEs in this paper have an exact solution contained in Table 2.

Price (PF)	PDE	$r^*{}_t\psi_t(\tau) = \delta + \frac{\partial\psi}{\partial t} - \frac{\partial\psi}{\partial\tau};\ \psi(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau} r^*(u)du} + \delta\int_t^{t+\tau} e^{-\int_t^s r^*(u)du}ds$
Valuation (PF)	PDE	$r_t v_t(\tau) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial\tau};\ v(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau} r(u)du} + \delta\int_t^{t+\tau} e^{-\int_t^s r(u)du}ds$
Price (default)	PDE	$\hat{r}^*{}_t\hat{\psi}_t(\tau) = \delta + \frac{\partial\hat{\psi}}{\partial t} - \frac{\partial\hat{\psi}}{\partial\tau} + \phi\mathbb{E}_t^X\left[\Phi\left(V\left[\hat{f}(\cdot,t)\right]\right)\psi(\tau,t,X_t) - \hat{\psi}_t(\tau)\right];\ v(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau}(\hat{r}^*(u)+\phi)du} + \int_t^{t+\tau} e^{-\int_t^s(\hat{r}^*(u)+\phi)du}\left(\delta + \phi\mathbb{E}_s^X\left[\Theta\left(V\left[\hat{f}(\cdot,t)\right]\right)\psi(\tau-s,t+s)\right]\right)ds$
Valuation (default)	PDE	$\hat{r}^*{}_t\hat{v}_t(\tau) = \delta + \frac{\partial\hat{v}}{\partial t} - \frac{\partial\hat{v}}{\partial\tau} + \phi\left(\mathbb{E}_s^X\left[(\Theta(V_t) + \Omega_t)\frac{U'(c_t)}{U'(\hat{c}_t)}v_t(\tau)\right] - \hat{v}_t(\tau)\right);\ \hat{v}(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau}(\hat{r}^*(s)+\phi)ds} + \int_t^{t+\tau} e^{-\int_t^s(\hat{r}^*(u)+\phi)du}\left(\delta + \phi\mathbb{E}_s^X\left[\left(\Omega(t+s) + \Theta\left(V\left[\hat{f}(\cdot,t+s)\right]\right)\right)\frac{U'(c(t+s))}{U'(\hat{c}(t+s))}v(\tau-s,t+s)\right]\right)ds$
Debt Profile	PDE	$\frac{\partial f}{\partial t} = \iota_t(\tau) + \frac{\partial f}{\partial\tau}$
	Integral	$f_t(\tau) = \int_\tau^{min\{T,\tau+t\}}\iota(s,t+\tau-s)ds + \mathbb{I}[T > t+\tau]\cdot f(0,\tau+t)$

Table 2: Equivalence between PDE and integral formulations.

B. Proofs

B.1 Proof of Proposition 1

Proof. Step 1. Solving the (post-shock) deterministic dynamics. First we analyze the deterministic transitional dynamics after the shock. This part of the proof mimics the one in . First, we construct a Lagrangian on the space of functions g such that are Lebesgue integrable, $\left\|e^{-\rho t/2}g_t(\tau)\right\|^2 < \infty$. The Lagrangian, after replacing c_t from the budget constraint, is:

$$\begin{aligned}\mathcal{L}[\iota, f] &= \int_0^\infty e^{-\rho t}U\left(y_t - f_t(0) + \int_0^T \left[q(\tau,t,\iota)\,\iota_t(\tau) - \delta f_t(\tau)\right]d\tau\right)dt \\ &\quad + \int_0^\infty \int_0^T e^{-\rho t} j_t(\tau)\left(-\frac{\partial f}{\partial t} + \iota_t(\tau) + \frac{\partial f}{\partial \tau}\right)d\tau dt,\end{aligned}$$

where $j_t(\tau)$ is the Lagrange multiplier associated to the law of motion of debt.

We consider a perturbation $h_t(\tau)$, $e^{-\rho t}h \in L^2([0,T]\times[0,\infty))$, around the optimal solution. Since the initial distribution f_0 is given, any feasible perturbation must satisfy $h_0(\tau) = 0$. In addition, we know that $f_t(T) = 0$ because $f_t(T^+) = 0$ (by construction) and issuances are infinitesimal. Thus, any admissible variation must also feature $h_t(T) = 0$. At an optimal solution f, the Lagrangian must satisfy $\mathcal{L}[\iota, f] \geq \mathcal{L}[\iota, f + \alpha h]$ for any perturbation $h_t(\tau)$.

Taking the derivative with respect to α —i.e., computing the Gâteaux derivative, for any suitable $h_t(\tau)$ we obtain:

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{L}[\iota, f+\alpha h]\Big|_{\alpha=0} &= \int_0^\infty e^{-\rho t}U'(c_t)\left[-h_t(0) - \int_0^T \delta h_t(\tau)\,d\tau\right]dt \\ &\quad - \int_0^\infty \int_0^T e^{-\rho t}\frac{\partial h}{\partial t} j_t(\tau)\,d\tau dt \\ &\quad + \int_0^\infty \int_0^T e^{-\rho t}\frac{\partial h}{\partial \tau} j_t(\tau)\,d\tau dt.\end{aligned}$$

We employ integration by parts to show that:

$$\begin{aligned}\int_0^\infty \int_0^T e^{-\rho t}\frac{\partial h}{\partial t} j_t(\tau)\, d\tau dt &= \int_0^T \int_0^\infty e^{-\rho t}\frac{\partial h}{\partial t} j_t(\tau)\, dt d\tau \\ &= \int_0^T \left(\lim_{s\to\infty} e^{-\rho s} h_s(\tau) j_s(\tau)] - h_0(\tau) j_0(\tau)\right) d\tau \\ &- \int_0^T \int_0^\infty e^{-\rho t}\left(\frac{\partial j_t(\tau)}{\partial t} - \rho j_t(\tau)\right) h_t(\tau)\, dt d\tau,\end{aligned}$$

and

$$\int_0^\infty e^{-\rho t}\int_0^T \frac{\partial h}{\partial \tau} j_t(\tau)\, d\tau dt \;=\; \int_0^\infty e^{-\rho t}\left[h_t(T) j_t(T) - h_t(0) j_t(0) - \int_0^T h_t(\tau)\frac{\partial j}{\partial \tau} d\tau\right] dt.$$

Replacing these calculations in the Lagrangian, and equating it to zero, yields:

$$\begin{aligned}0 \;=\; &\int_0^\infty e^{-\rho t} U'(c_t)\left[-h_t(0) - \int_0^T \delta h_t(\tau)\, d\tau\right] dt \\ &+ \int_0^\infty \int_0^T e^{-\rho t}\left(-\rho j - \frac{\partial j}{\partial \tau} + \frac{\partial j}{\partial t}\right) h_t(\tau)\, d\tau dt \\ &+ \int_0^\infty e^{-\rho t}\left(h_t(T) j_t(T) - h_t(0) j_t(0)\right) dt \\ &- \int_0^\infty \lim_{s\to\infty} e^{-\rho s} h_s(\tau) j_s(\tau)\, d\tau + h_0(\tau) j_0(\tau).\end{aligned}$$

We rearrange terms to obtain:

$$\begin{aligned}0 \;=\; &-\int_0^\infty e^{-\rho t}\left[U'(c_t) - j_t(0)\right] h_t(0)\, dt \\ &+ \int_0^\infty \int_0^T e^{-\rho t}\left(-\rho j - U'(c)\,\delta - \frac{\partial j}{\partial \tau} + \frac{\partial j}{\partial t}\right) h_t(\tau)\, d\tau dt \\ &- \int_0^\infty e^{-\rho t}\left(h_t(T) j_t(T)\right) dt \\ &- \int_0^\infty \lim_{s\to\infty} e^{-\rho s} h_s(\tau) j_s(\tau)\, d\tau + h_0(\tau) j_0(\tau).\end{aligned} \tag{23}$$

Since $h_t(T) = h_0(\tau) = 0$ is a condition for any admissible variation, then, both the third line in equation (23) and the second term in the fourth line are equal to zero. Furthermore, because (23) needs to hold for any feasible variation $h_t(\tau)$, all the terms that multiply $h_t(\tau)$ should equal zero. The latter, yields a system of necessary conditions for

the Lagrange multipliers:

$$\begin{aligned} \rho j_t(\tau) &= -\delta U'(c_t) + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau}, \text{ if } \tau \in (0,T], \\ j_t(0) &= -U'(c_t), \text{ if } \tau = 0, \\ \lim_{t\to\infty} e^{-\rho t} j_t(\tau) &= 0, \text{ if } \tau \in (0,T]. \end{aligned} \tag{24}$$

Next, we perturb the control. We proceed in a similar fashion:

$$\begin{aligned} \tfrac{d}{d\alpha}\mathcal{L}[\iota + \alpha h, , f]\big|_{\alpha=0} = & \int_0^\infty e^{-\rho t} U'(c_t) \left[\int_0^T \left(\frac{\partial q}{\partial \iota} \iota_t(\tau) + q_t(\tau, \iota) \right) h_t(\tau)\, d\tau \right] dt \\ & + \int_0^\infty \int_0^T e^{-\rho t} h_t(\tau)\, j_t(\tau)\, d\tau dt. \end{aligned}$$

Collecting terms and setting the Lagrangian to zero, we obtain:

$$\int_0^\infty \int_0^T e^{-\rho t} \left[j_t(\tau) + U'(c_t) \left(\frac{\partial q}{\partial \iota} \iota_t(\tau) + q_t(\tau, \iota) \right) \right] h_t(\tau)\, d\tau dt = 0.$$

Thus, setting the term in parenthesis to zero, amounts to setting:

$$U'(c_t) \left(\frac{\partial q}{\partial \iota} \iota_t(\tau) + q_t(\tau, \iota) \right) = -j_t(\tau). \tag{25}$$

Next, we define the Lagrange multiplier in terms of goods:

$$v_t(\tau) = -j_t(\tau) / U'(c_t). \tag{26}$$

Taking the derivative of $v_t(\tau)$ with respect to t and τ we can express the necessary conditions, (24) in terms of v. In particular, we transform the PDE in (24) into the summary equations in the Proposition. That is:

$$\begin{aligned} \left(\rho - \frac{U''(c_t)\, c_t}{U'(c_t)} \frac{\dot{c}_t}{c_t} \right) v_t(\tau) &= \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0,T], \\ v_t(0) &= 1, \text{ if } \tau = 0, \\ \lim_{t\to\infty} e^{-\rho t} v_t(\tau) &= 0, \text{ if } \tau \in (0,T]; \end{aligned}$$

and the first-order condition, (25), is now given by:

$$\frac{\partial q}{\partial \iota}\iota_t(\tau) + q_t(\tau, \iota) = v_t(\tau)$$

as we intended to show.

Step 2. Computing the Gateaux derivative of the value functional. Next, we show that

$$\frac{d}{d\alpha} V[f_0(\cdot) + \alpha h_0(\cdot)]|_{\alpha=0} = \int_0^T j_0(\tau) h_0(\tau) d\tau,$$

where

$$V[f_0(\cdot)] = \int_0^\infty e^{-\rho t} U(c_t) dt.$$

To avoid confusions, we denote by (ι^*, f^*) the optimal issuance policy and debt distributions. First, note that

$$V[f_0(\cdot)] = \mathcal{L}[\iota^*, f^*].$$

This follows from the fact that

$$\begin{aligned}
V[f(\cdot,0)] &= \int_0^\infty e^{-\rho t} U\left(y_t - f_t^*(0) + \int_0^T [q_t(\tau, \iota^*)\iota_t^*(\tau) - \delta f_t^*(\tau)] d\tau\right) dt \\
&= \int_0^\infty e^{-\rho t} U\left(y_t - f_t^*(0) + \int_0^T [q(\tau, t, \iota^*)\iota_t^*(\tau) - \delta f_t^*(\tau)] d\tau\right) dt \\
&+ \int_0^\infty \int_0^T e^{-\rho t} j_t^*(\tau)\left(-\frac{\partial f^*}{\partial t} + \iota_t^*(\tau) + \frac{\partial f^*}{\partial \tau}\right) d\tau dt, \\
&= \mathcal{L}[\iota^*, f^*].
\end{aligned}$$

where the first line is the definition of $V[f(\cdot,0)]$, the second line is the fact that

$$-\frac{\partial f^*}{\partial t} + \iota_t^*(\tau) + \frac{\partial f^*}{\partial \tau} = 0$$

for for every τ, t and the last line is the definition of the Lagrangian. Next we compute $\frac{\partial}{\partial \alpha}\mathcal{L}[\iota^*, f^* + \alpha h_0(\tau)]\big|_{\alpha=0}$. The derivative with respect to a general variation $h(\tau, t)$

is given by:

$$\begin{aligned}\frac{\partial}{\partial\alpha}\mathcal{L}\left[\iota, f+\alpha h\right]\Big|_{\alpha=0} &= \int_0^\infty e^{-\rho t}U'\left(c_t\right)\left[-h_t\left(0\right)-\int_0^T \delta h_t\left(\tau\right)d\tau\right]dt \\ &\quad -\int_0^\infty\int_0^T e^{-\rho t}\frac{\partial h}{\partial t}j_t\left(\tau\right)d\tau dt \\ &\quad +\int_0^\infty\int_0^T e^{-\rho t}\frac{\partial h}{\partial \tau}j_t\left(\tau\right)d\tau dt.\end{aligned} \tag{27}$$

We employ integration by parts to show that:

$$\begin{aligned}\int_0^\infty\int_0^T e^{-\rho t}\frac{\partial h}{\partial t}j_t\left(\tau\right)d\tau dt &= \int_0^T\left(\lim_{s\to\infty}e^{-\rho s}[h_s\left(\tau\right)j_s\left(\tau\right)]-h_0\left(\tau\right)j_0\left(\tau\right)\right)d\tau \\ &\quad -\int_0^T\int_0^\infty e^{-\rho t}\left(\frac{\partial j_t\left(\tau\right)}{\partial t}-\rho j_t\left(\tau\right)\right)h(\tau,t)dtd\tau\end{aligned}$$

and

$$\int_0^\infty e^{-\rho t}\int_0^T\frac{\partial h}{\partial\tau}j_t\left(\tau\right)d\tau dt = \int_0^\infty e^{-\rho t}\left[h\left(T,t\right)j\left(T,t\right)-h_t\left(0\right)j_t\left(0\right)-\int_0^T h_t(\tau)\frac{\partial j}{\partial\tau}d\tau\right]dt.$$

Note that we have not yet used optimality. Using the particular case of interest, i.e.:

$$h_0(\tau)=h_t(\tau)\delta_t,$$

where δ_t is the Dirac delta, and plugging it in equation (27):

$$\begin{aligned}\frac{\partial}{\partial\alpha}\mathcal{L}\left[\iota^*, f^*+\alpha h_0(\cdot)\right]\Big|_{\alpha=0} &= U'\left(c_0^*\right)\left[-h_0(0)-\int_0^T\delta h_t\left(\tau\right)d\tau\right] \\ &\quad +\int_0^T h_t\left(\tau\right)j_t\left(\tau\right)d\tau \\ &\quad +\int_0^T\left(\frac{\partial j_t\left(\tau\right)}{\partial t}-\rho j_t\left(\tau\right)\right)h(\tau,0)d\tau, \\ &\quad -h_t\left(0\right)j_t\left(0\right)-\int_0^T h_0(\tau)\frac{\partial j}{\partial\tau}d\tau dt\end{aligned}$$

Because (ι^*, f^*) is an optimum, we know that for all $\tau\in(0,T]$ the following holds

$$\frac{\partial j_t\left(\tau\right)}{\partial t}-\rho j_t\left(\tau\right)-\frac{\partial j_t\left(\tau\right)}{\partial\tau}-\delta=0$$

and for $\tau = 0$ it also holds that $U'(c_0^*) = -j_t(0)$. This implies that

$$\frac{\partial}{\partial \alpha}\mathcal{L}\left[\iota^*, f^* + \alpha h_0(\cdot)\right]\Big|_{\alpha=0} = \int_0^T h_0(\tau)\, j_0(\tau)\, d\tau.$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial \alpha}\mathcal{L}\left[\iota^*, f^* + \alpha h_0(\cdot)\right]\Big|_{\alpha=0} &= \int_0^T h_0(\tau)\, j_0(\tau)\, d\tau \\ &= \frac{\partial}{\partial \alpha} V\left[f_0(\cdot) + \alpha h_0(\cdot)\right].\end{aligned}$$

Step 2.1 Setting the Lagrangian before the shock arrival. Let $\mathbb{V}\left[\hat{f}_{t^o}(\cdot), X_{t^o}\right]$ denote the expected value of the government, at the instant t^o where the option to default is available, but prior to the decision of default. This value equals:

$$\mathbb{V}\left[\hat{f}_{t^o}(\cdot), X_{t^o}\right] = \mathbb{E}_{t^o}^X\left[\underbrace{\Gamma\left(V\left[\hat{f}_{t^o}(\cdot)\right], X_{t^o}\right)}_{\text{Default}} + \underbrace{\Theta\left(V\left[\hat{f}_{t^o}(\cdot), X_{t^o}\right]\right) V\left[\hat{f}_{t^o}(\cdot), X_{t^o}\right]}_{\text{No default}}\right],$$

where the first term in the expectation is the expected utility conditional on default given by $\Gamma(x) \equiv \int_x^\infty z d\Theta(z)$. The second term is the probability of no default time the perfect-foresight value. The Lagrangian is:[22]

$$\begin{aligned}\mathcal{L}\left[\hat{\iota}, \hat{f}, \hat{\psi}\right] &= \mathbb{E}_{t^o}^X\left[\int_0^{t^o} e^{-\rho s} U(\hat{c}_s)\, ds + e^{-\rho t^o}\mathbb{V}\left[\hat{f}_{t^o}(\cdot), X_{t^o}\right]\right. \\ &\quad + \int_0^{t^o}\int_0^T e^{-\rho s}\hat{\jmath}_s(\tau)\left(-\frac{\partial \hat{f}}{\partial s} + \hat{\iota}_s(\tau) + \frac{\partial \hat{f}}{\partial \tau}\right) d\tau ds \\ &\quad \left. + \int_0^{t^o}\int_0^T e^{-\rho s}\hat{\mu}_s(\tau)\left(-\hat{r}^*(s)\,\hat{\psi}_s(\tau) + \delta + \frac{\partial \hat{\psi}}{\partial s} - \frac{\partial \hat{\psi}}{\partial \tau}\right) d\tau ds\right].\end{aligned}$$

In the Lagrangian, $\mathbb{E}^{t^o}$ denotes the conditional expectation with respect to the random time t^o. Here $\hat{\jmath}_s(\tau)$ and $\hat{\mu}_s(\tau)$ are the Lagrange multipliers. The first set of multipliers, $\hat{\jmath}_s(\tau)$, are associated with the law of motion of debt. The second set of multipliers, $\hat{\mu}_s(\tau)$, are associated with the law of motion of bond prices. These terms appear be-

[22] The Lagragian should be defined on a suitable space of functions, typically a Sobolev space. We abstract from these technical details.

cause the government understands how its influence on the maturity profile affects the incentives to default, and hence impacts bond prices. This happens through the terminal condition:

$$\begin{aligned}\hat{\psi}_{t^o}(\tau) &= \mathbb{E}_{t^o}^X\left\{\Theta\left(V\left[\hat{f}_{t^o}(\cdot), X_{t^o}\right]\right)\psi_{t^o}(\tau)\right\},\\ \hat{\psi}_t(0) &= 1.\end{aligned}$$

The terminal condition reflects that, at date t^o, the bond price is zero if default occurs. Otherwise it equals the perfect-foresight price, $\psi_{t^o}(\tau)$, if default does not occur.

Step 2.2. Re-writing the Lagrangian. Proceeding as in the proof of the deterministic case, as an intermediate step we integrate by parts the terms that involve time or maturity derivatives of $\hat{f}$ and $\hat{\psi}$. The Lagrangian $\mathcal{L}\left[\hat{\imath}, \hat{f}, \hat{\psi}\right]$ can thus be expressed as:

$$\begin{aligned}&\mathbb{E}_{t^o}^X\left[\int_0^{t^o} e^{-\rho s}U(\hat{c}_s)\,ds + e^{-\rho t^o}\mathbb{V}\left[\hat{f}_{t^o}(\cdot), X_{t^o}\right] - \int_0^T e^{-\rho t^o}\hat{f}_{t^o}(\tau)\,\hat{\jmath}_{t^o}(\tau)d\tau\right.\\
&+\int_0^T \hat{f}_0(\tau)\,\hat{\jmath}_0(\tau)\,d\tau + \int_0^{t^o}\int_0^T e^{-\rho s}\hat{f}_s(\tau)\left(\frac{\partial \hat{\jmath}}{\partial s} - \rho\hat{\jmath}_s(\tau)\right)ds d\tau\\
&+\int_0^{t^o} e^{-\rho s}\hat{f}_s(T)\,\hat{\jmath}_s(T)\,ds - \int_0^{t^o} e^{-\rho s}\hat{f}_0(s)\,\hat{\jmath}_0(0)\,ds\\
&-\int_0^{t^o}\int_0^T e^{-\rho s}\hat{f}_s(\tau)\frac{\partial \hat{\jmath}}{\partial \tau}d\tau ds + \int_0^{t^o}\int_0^T e^{-\rho s}\hat{\jmath}_s(\tau)\,\hat{\imath}_s(\tau)\,d\tau ds\\
&+\int_0^{t^o}\int_0^T e^{-\rho s}\hat{\mu}_s(\tau)\left(-\hat{r}^*(s)\,\hat{\psi}_s(\tau) + \delta\right)d\tau ds,\\
&+\int_0^T\left[e^{-\rho t^o}\hat{\mu}_{t^o}(\tau)\,\hat{\psi}_{t^o}(\tau) - \hat{\mu}_0(\tau)\,\hat{\psi}_0(\tau)\right]d\tau\\
&-\int_0^{t^o}\int_0^T e^{-\rho s}\hat{\psi}_s(\tau)\left(\frac{\partial \hat{\mu}}{\partial s} - \rho\hat{\mu}_s(\tau)\right)d\tau ds\\
&-\int_0^{t^o} e^{-\rho s}\left[\hat{\mu}_s(T)\,\hat{\psi}_s(T) - e^{-\rho s}\hat{\mu}_s(0)\,\hat{\psi}_s(0)\right]ds\\
&\left.+\int_0^{t^o}\int_0^T e^{-\rho s}\hat{\psi}_s(\tau)\frac{\partial \hat{\mu}}{\partial \tau}d\tau ds\right].\end{aligned}$$

Step 2.3. Computing expectations. If we group terms, substitute the terminal conditions $f_s(T) = 0$ and $\hat{\psi}_{t^o}(\tau) = \Theta\left(V\left[\hat{f}_{t^o}(\cdot), X_{t^o}\right]\right)\psi_{t^o}(\tau)$ and compute the expected value

with respect to t^o, we can express the Lagrangian $\mathcal{L}\left[\hat{\iota},\hat{f},\hat{\psi}\right]$ as:

$$
\begin{aligned}
&\int_0^\infty e^{-(\rho+\phi)s}U\left(\hat{c}_s\right)ds \\
&-\int_0^\infty e^{-(\rho+\phi)s}\hat{f}_0\left(s\right)\hat{\jmath}_s\left(0\right)ds \\
&-\int_0^\infty e^{-(\rho+\phi)s}\left[\hat{\mu}_s\left(T\right)\hat{\psi}_s\left(T\right)-\mu_s\left(0\right)\hat{\psi}_s\left(0\right)\right]ds \\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{f}_s\left(\tau\right)\left(\frac{\partial\hat{\jmath}}{\partial s}-\rho\hat{\jmath}_s\left(\tau\right)\right)dsd\tau \\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{f}_s\left(\tau\right)\frac{\partial\hat{\jmath}}{\partial\tau}d\tau ds \\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{\jmath}_s\left(\tau\right)\hat{\iota}_s\left(\tau\right)d\tau ds \\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{\mu}_s\left(\tau\right)\left(-\hat{r}_s^*\hat{\psi}_s(\tau)+\delta\right)d\tau ds \\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{\psi}_s(\tau)\left(\frac{\partial\hat{\mu}}{\partial s}-\rho\hat{\mu}_s(\tau)\right)d\tau ds \\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{\psi}_s\left(\tau\right)\frac{\partial\hat{\mu}}{\partial\tau}d\tau ds \\
&+\int_0^T f_0\left(\tau\right)\hat{\jmath}_0\left(\tau\right)d\tau-\int_0^T\hat{\mu}_0\left(\tau\right)\hat{\psi}_0\left(\tau\right)d\tau \\
&+\int_0^\infty e^{-(\rho+\phi)s}\phi\mathbb{V}\left[\hat{f}_s\left(\cdot\right),X_s\right]ds \\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi\hat{f}_s\left(\tau\right)\hat{\jmath}_s\left(\tau\right)d\tau ds \\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi\hat{\mu}_s\left(\tau\right)\mathbb{E}_s^X\left\{\theta\left(V\left[\hat{f}_s\left(\cdot\right),X_s\right]\right)\psi_s(\tau)\right\}d\tau ds.
\end{aligned}
$$

Next, we compute the Gâteaux derivatives with respect to each of the three arguments of the value function at a time.

Step 3. Computing the derivatives. Computing the derivatives requires two steps. *Step 3.1. Gâteaux derivative with respect to the issuances.* If we consider a perturbation around issuances and equalize it to zero, $\frac{\partial}{\partial\alpha}\mathcal{L}\left[\hat{\iota}+\alpha h,\hat{f},\hat{\psi}\right]\Big|_{\alpha=0}=0$, the result is identical to the risk-less case:

$$
U'\left(\hat{c}_t\right)\left(\frac{\partial q}{\partial\iota}\hat{\iota}_t\left(\tau\right)+q\left(t,\tau,\hat{\iota}\right)\right)=-\hat{\jmath}_t\left(\tau\right).
$$

Step 3.2. Gâteaux derivative with respect to the debt density. Since the distribution at the beginning $f_0(\tau)$ is given, any feasible perturbation must feature $h_0(\tau) = 0$ for any $\tau \in (0, T]$. In addition, we know that $h_t(T) = 0$, because $f_t(T) = 0$. The Gâteaux derivative of the continuation value with respect to the debt density is:

$$\frac{d}{d\alpha} \mathbb{V}\left[\hat{f}_s(\cdot) + \alpha h_s(\cdot), X_s\right]\Big|_{\alpha=0} = \mathbb{E}_s^X\left\{\Theta\left(V\left[\hat{f}_s(\cdot), X_s\right]\right)\int_0^T j_s(\tau) h_s(\tau) d\tau\right\},$$

where we have taken into account the fact that $\frac{d}{dx}(\Gamma(x) + \Theta(x) x) = \Theta(x)$ and $--$:

$$\frac{d}{d\alpha} V\left[\hat{f}_s(\cdot) + \alpha h_s(\cdot)\right]\Big|_{\alpha=0} = \int_0^T j_s(\tau) h_s(\tau) d\tau.$$

Similarly, the Gâteaux derivative of the terminal bond price with respect to the debt density is

$$\frac{d}{d\alpha} \mathbb{E}_s^X\left\{\Theta\left(V\left[\hat{f}_s(\cdot) + \alpha h_s(\cdot), X_s\right]\right)\psi_s(\tau)\right\}\Big|_{\alpha=0} = \ldots$$
$$\mathbb{E}_s^X\left\{\theta\left(V\left[\hat{f}_s(\cdot), X_s\right]\right)\psi_s(\tau)\int_0^T j_s(\tau') h_s(\tau') d\tau'\right\},$$

where $\theta(x) \equiv \frac{d}{dx}\Theta(x)$ is the probability density. The Gâteaux derivative of the La-

grangian with respect to the debt density, $\frac{d}{d\alpha}\mathcal{L}\left[\hat{\iota},\hat{f}+\alpha h,\hat{\psi}\right]\Big|_{\alpha=0}$, is thus:

$$\begin{aligned}
&\int_0^\infty e^{-(\rho+\phi)s}U'\left(\hat{c}_s\right)\left[-h_s\left(0\right)+\int_0^T\left(-\delta\right)h_s\left(\tau\right)d\tau\right]ds\\
&-\int_0^\infty e^{-(\rho+\phi)s}h_s\left(0,s\right)\hat{\jmath}_s\left(0\right)ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}h_s\left(\tau\right)\left(\frac{\partial\hat{\jmath}}{\partial s}-\rho\hat{\jmath}_s\left(\tau\right)\right)dsd\tau\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}h_s\left(\tau\right)\frac{\partial\hat{\jmath}}{\partial\tau}d\tau ds\\
&+\int_0^T h_0\left(\tau\right)\hat{\jmath}_0\left(\tau\right)d\tau\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\theta\mathbb{E}_s^X\left\{\Theta\left(V\left[\hat{f}_s\left(\cdot\right),X_s\right]\right)j_s\left(\tau\right)\right\}h_s\left(\tau\right)d\tau ds\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi h_s\left(\tau\right)\hat{\jmath}_s\left(\tau\right)d\tau ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi\mu_s\left(m\right)\mathbb{E}_s^X\left\{\theta\left(V\left[\hat{f}_s\left(\cdot\right),X_s\right]\right)\psi_s(m)\int_0^T j_s\left(\tau\right)h_s\left(\tau\right)d\tau\right\}dmds.
\end{aligned}$$

The value of the Gâteaux derivative of the Lagrangian for any perturbation, must be zero, i.e. $\frac{d}{d\alpha}\mathcal{L}\left[\hat{\iota},\hat{f}+\alpha h,\hat{\psi}\right]\Big|_{\alpha=0}=0$. Thus, a necessary condition is that all terms that multiply any entry of $h_s\left(\tau\right)$ add up to zero. We summarize the necessary conditions into:

$$\begin{aligned}
\rho\hat{\jmath}_s\left(\tau\right) &= \left(-\delta\right)U'\left(\hat{c}\left(s\right)\right)+\frac{\partial\hat{\jmath}}{\partial s}-\frac{\partial\hat{\jmath}}{\partial\tau} &(28)\\
&+ \phi\mathbb{E}_s^X\left\{\left[\Theta\left(V\left[\hat{f}_s\left(\cdot\right),X_s\right]\right)+\theta\left(V\left[\hat{f}_s\left(\cdot\right),X_s\right]\right)\int_0^T\hat{\mu}_s\left(m\right)\psi_s(m)dm\right]j_s\left(\tau\right)-\hat{\jmath}_s\left(\tau\right)\right\} &(29)\\
\hat{\jmath}_s\left(0\right) &= -U'\left(\hat{c}_s\right).
\end{aligned}$$

Step 3.3. Gâteaux derivative with respect to the bond price. In the case of the Gâteaux derivatives with respect to the evolution of the price $\hat{\psi}$, $\frac{d}{d\alpha}\mathcal{L}\left[\hat{\iota},f,\hat{\psi}+\alpha h\right]\Big|_{\alpha=0}$, we need to work first with the Lagragian before expectations have been computed. The reason is the following: only bonds that mature after default can be affected by the government's policies and hence the variations have to be zero for those bonds that mature before default, $\tilde{h}_t\left(\tau\right)=0$, if $\tau+t<t^o$. To incorporate this, we assume that admissible

perturbations are of the form $\tilde{h}_t(\tau) = h_t(\tau) 1_{\{\tau+t\geq t^o\}}$, where $h_t(\tau)$ is unrestricted. The Gâteaux derivative is then

$$\begin{aligned}
&\mathbb{E}_{t^o}^X\left[\int_0^{t^o} e^{-\rho s}U'(\hat{c}_s)\left(\int_0^T \hat{\iota}_s(\tau)\frac{\partial q}{\partial \hat{\psi}}1_{\{\tau+s\geq t^o\}}h_s(\tau)\,d\tau\right)ds\right.\\
&+\int_0^{t^o}\int_0^T e^{-\rho s}\hat{\mu}_s(\tau)\left(-\hat{r}_s^* 1_{\{\tau+s\geq t^o\}}h_s(\tau)\right)d\tau ds\\
&-\int_0^T \hat{\mu}_0(\tau)1_{\{\tau\geq t^o\}}h_0(\tau)\,d\tau\\
&-\int_0^{t^o}\int_0^T e^{-\rho s}1_{\{\tau+s\geq t^o\}}h_s(\tau)\left(\frac{\partial\hat{\mu}}{\partial s}-\rho\hat{\mu}_s(\tau)\right)d\tau ds\\
&-\int_0^{t^o} e^{-\rho s}\left[1_{\{T+s\geq t^o\}}h_s(T)\hat{\psi}_s(T)-e^{-\rho s}1_{\{s\geq t^o\}}h_s(0)\hat{\psi}_s(0)\right]ds\\
&\left.+\int_0^{t^o}\int_0^T e^{-\rho s}1_{\{\tau+s\geq t^o\}}h_s(\tau)\frac{\partial\hat{\mu}}{\partial\tau}d\tau ds\right].
\end{aligned}$$

Note that the perturbation is only around $\hat{\psi}_s(\tau)$ and not $\psi_s(\tau)$, the terminal price after default, which is given. Since at maturity, bonds have a value of 1, $h_s(0)=0$, because no perturbation can affect that price. If we compute the expectation with respect to the random arrival time, t^o, we get:

$$\begin{aligned}
&\int_0^\infty e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)U'(\hat{c}_s)\left[\int_0^T\hat{\iota}_s(\tau)\frac{\partial q}{\partial\hat{\psi}}h_s(\tau)\,d\tau\right]ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)\hat{\mu}_s(\tau)\left(-\hat{r}_s^*h_s(\tau)\right)d\tau ds\\
&-\int_0^T\left(1-e^{-\phi\tau}\right)\mu_0(\tau)h_0(\tau)\,d\tau\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)h(\tau,s)\left(\frac{\partial\hat{\mu}}{\partial s}-\rho\mu_s(\tau)\right)d\tau ds\\
&-\int_0^\infty e^{-(\rho+\phi)s}\left(1-e^{-\phi T}\right)\mu_s(T)h_s(T)\,ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)\hat{\psi}_s(\tau)\frac{\partial\hat{\mu}}{\partial\tau}d\tau ds,
\end{aligned}$$

where we use

$$\mathbb{E}_{t^o}^X\left[1_{\{\tau+s>t^o>s\}}\right]=e^{-\phi s}\left(1-e^{-\phi\tau}\right).$$

Again, as the Gâteaux derivative should be zero for any suitable $h(\tau, s)$, the optimality condition is

$$\begin{aligned}
\left(\hat{r}_s^* - \rho\right)\hat{\mu}_s(\tau) &= U'\left(\hat{c}_s\right)\hat{\imath}_s\left(\tau\right)\frac{\partial q}{\partial \hat{\psi}} - \frac{\partial \hat{\mu}}{\partial s} + \frac{\partial \hat{\mu}}{\partial \tau}, \\
\hat{\mu}_s\left(T\right) &= 0, \\
\hat{\mu}_0\left(\tau\right) &= 0.
\end{aligned}$$

The solution to this PDE is

$$\hat{\mu}_s\left(\tau\right) = \int_{\max\{s+\tau-T,0\}}^{s} e^{-\int_z^s (\hat{r}^*(u)-\rho)du} U'\left(\hat{c}_z\right)\hat{\imath}\left(\tau+s-z,z\right)\frac{\partial q_z}{\partial \hat{\psi}}\left(\tau+s-z\right)dz.$$

If we integrate the discount factor of the government with respect to time, we obtain the following identity:

$$\int_z^s \hat{r}_u du = \int_z^s \rho du - \int_z^s \frac{U''\left(\hat{c}_u\right)}{U'\left(\hat{c}_u\right)}\hat{c}_u \frac{\dot{\hat{c}}_u}{\hat{c}_u} du.$$

Therefore, we have that

$$\int_z^s \hat{r}_u du - \int_z^s \rho du = -\log\left(U'\left(\hat{c}_u\right)\right)\Big|_z^s.$$

We obtain the following identity:

$$e^{\int_z^s \rho du} = e^{\int_z^s \hat{r}_u du}\frac{U'\left(\hat{c}_s\right)}{U'\left(\hat{c}_z\right)}.$$

Thus, the PDE for $\hat{\mu}_s\left(\tau\right)$ can be written as:

$$\hat{\mu}_s\left(\tau\right) = U'\left(\hat{c}_s\right)\int_{\max\{s+\tau-T,0\}}^{s} e^{-\int_z^s (\hat{r}_u^* - \hat{r}_u)du}\hat{\imath}_z\left(\tau+s-z\right)\frac{\partial q_z}{\partial \hat{\psi}}\left(\tau+s-z\right)dz.$$

Notice that

$$\begin{aligned}\hat{\imath}\frac{\partial q}{\partial \hat{\psi}} &= \left(\hat{\imath}-\frac{\bar{\lambda}}{2}\left(\hat{\imath}\right)^{2}\right)\\ &= \frac{1}{2\bar{\lambda}}\left(\frac{\hat{\psi}^{2}-\hat{v}^{2}}{\hat{\psi}^{2}}\right)\\ &= \frac{1}{2\bar{\lambda}}\left(1-\frac{\hat{v}^{2}}{\hat{\psi}^{2}}\right).\end{aligned}$$

where the second line uses the optimal issuance rule. Hence, we can write the price multiplier as:

$$\hat{\mu}_{s}\left(\tau\right)=U'\left(\hat{c}_{s}\right)\int_{\max\{s+\tau-T,0\}}^{s}e^{-\int_{z}^{s}(\hat{r}_{u}^{*}-\hat{r}_{u})du}\frac{1}{2\bar{\lambda}}\left(1-\left(\frac{\hat{v}_{z}\left(\tau+s-z\right)}{\hat{\psi}_{z}\left(\tau+s-z\right)}\right)^{2}\right)dz.$$

We employ this solution in the main text.

Step 4: From Lagrange multipliers to valuations. We now employ the definitions of $\hat{v}_{s}\left(\tau\right)=-\hat{\jmath}_{s}\left(\tau\right)/U'\left(\hat{c}_{s}\right)$ and $v_{s}\left(\tau\right)=-j_{s}\left(\tau\right)/U'\left(\hat{c}_{s}\right)$, we can express equations (28)-(29) as

$$\begin{aligned}\hat{r}_{s}\hat{v}_{s}\left(\tau\right) &= \delta+\frac{\partial\hat{v}}{\partial s}-\frac{\partial\hat{v}}{\partial\tau}\\ &\quad+\phi\mathbb{E}_{s}^{X}\left\{\left[\Theta\left(V_{s}\right)+\theta\left(V_{s}\right)\int_{0}^{T}\hat{\mu}_{s}\left(m\right)\psi_{s}(m)dm\right]\frac{U'\left(c_{s}\right)}{U'\left(\hat{c}_{s}\right)}v_{s}\left(\tau\right)-\hat{v}_{s}\left(\tau\right)\right\},\\ \hat{v}_{s}\left(0\right) &= 1,\end{aligned}$$

where we use the notation $\Theta\left(V_{s}\right)\equiv\Theta\left(V\left[\hat{f}_{s}\left(\cdot\right),X_{s}\right]\right)$ and $\theta\left(V\left(s\right)\right)\equiv\theta\left(V\left[\hat{f}_{s}\left(\cdot\right),X_{s}\right]\right)$. Therefore valuations can be expressed as

$$\begin{aligned}\hat{v}_{t}\left(\tau\right) &= e^{-\int_{t}^{t+\tau}(\hat{r}_{s}+\phi)ds}\\ &\quad+\phi\int_{t}^{t+\tau}e^{-\int_{t}^{s}(\hat{r}_{u}+\phi)du}\left(\delta+\mathbb{E}_{s}^{X}\left[\left(\Theta\left(V_{t+s}\right)+\Omega_{t+s}\right)\frac{U'\left(c_{t+s}\right)}{U'\left(\hat{c}_{t+s}\right)}v_{t+s}\left(\tau-s\right)\right]\right)ds,\end{aligned}$$

where

$$\Omega_{t}=\theta\left(V_{t}\right)\int_{0}^{T}\hat{\mu}_{t}\left(m\right)\psi_{t}(m)dm.$$

□

B.2 Proof of Proposition 2

Preliminary Calculations. From , Lemma 1, we know that the elasticity of the WAM with respect to a parameter θ, $\epsilon^{\mu}_{t,\theta} \equiv \frac{\partial \mu_t}{\partial \theta} \cdot \frac{\theta}{\mu_t}$, is related to the elasticity of the issuance at a given maturity, $\epsilon^{\tau}_{t,\theta} \equiv \frac{\partial \iota_t(\tau)}{\partial \theta} \cdot \frac{\theta}{\iota_t(\tau)}$, as established in the following lemma.

Lemma 1. *[**Monotone Comparative Statics**] Assume that all issuances are positive, $\iota_t(\tau) > 0$. If the issuance elasticities, $\epsilon^{\tau}_{t,\theta}$, are increasing (decreasing) in maturity τ, $\frac{d\epsilon^{\tau}_{t,\theta}}{d\tau} > 0$, for all $\tau \in [0,T]$, then the elasticity of the WAM with respect to θ, $\epsilon^{\mu}_{t,\theta}$, is positive (negative).*

Consider a RSS with $\delta = 0$ and $\bar{\lambda} \to \infty$. Note that in this case:

$$\lim_{\bar{\lambda}\to\infty} f_{ss}(\tau) = 0, \quad \lim_{\bar{\lambda}\to\infty} \iota_t(\tau) = 0,$$

The RSS valuations are given by:

$$\rho \hat{v}_{rss}(\tau) = -\frac{\partial \hat{v}_{rss}}{\partial \tau} + \phi \cdot \left(\mathbb{E}^{X}_{rss}\left[(\Theta_0 + \Omega_0) U'(y_0/y_{ss}) v_0(\tau) \right] - \hat{v}_{rss}(\tau) \right), \ \hat{v}_{rss}(0) = 1.$$

We can verify the following Lemma.

Lemma 2. *The solution to $\hat{v}_{rss}(\tau)$ is:*

$$\hat{v}_{rss}(\tau) = \exp\left(-(\rho+\phi)\tau\right) + \phi \int_0^{\tau} \exp\left(-(\rho+\phi)(\tau - s)\right) \mathbb{E}^{X}_{rss}\left[(\Theta_0 + \Omega_0) U'(y_0/y_{ss}) v_0(s) \right] ds.$$

Proof. We can verify the solution since:

$$(\rho+\phi) \Bigg(\underbrace{\exp\left(-(\rho+\phi)\tau\right) - (\rho+\phi)\phi \int_0^{\tau} \exp\left(-(\rho+\phi)(\tau-s)\right) \mathbb{E}^{X}_{rss}\left[(\Theta_{rss} + \Omega_{rss}) U'(y_0/y_{ss}) v_t(s) \right] ds}_{\hat{v}_{rss}(\tau)} \Bigg)$$

$$= -\frac{\partial \hat{v}_{rss}}{\partial \tau} + \phi \mathbb{E}^{X}_{rss}\left[(\Theta_0 + \Omega_0) U'(y_0/y_{ss}) v_0(\tau) \right].$$

Thus,

$$-\frac{\partial \hat{v}_{rss}}{\partial \tau} = (\rho+\phi)\hat{v}_{rss}(\tau) - \phi \mathbb{E}^{X}_{rss}\left[(\Theta_0 + \Omega_0) v_0(\tau) U'(y_0/y_{ss}) \right].$$

□

Next, We consider a single event of a jump in income and investigate the special limit case $\alpha \to \infty$:

$$\lim_{\{\bar{\lambda},\alpha\}\to\infty} c_t = y_t = y_{ss} \; \forall t \neq t^o.$$

At t^o there is an instantaneous shock. Thus, y_t is characterized as a single impulse, and thus, a small jump in marginal utility $U'(y_0/y_{ss})$. Since, income is steady and equal to consumption for all periods other than t^o, we obtain:

$$v_0(\tau) = \exp(-\rho\tau).$$

We prove the following auxiliary Lemma.

Lemma 3. *The solution to* $\hat{v}_{rss}(\tau)$ *is:*

$$\lim_{\bar{\lambda}\to\infty}\lim_{\alpha\to\infty}\left[\hat{v}_{rss}(\tau)\right] = \exp(-(\rho+\phi)\tau)\left(1+\mathbb{E}^X_{rss}\left[(\Theta_0+\Omega_0)U'(y_0/y_{ss})\right](\exp(\phi\tau)-1)\right).$$

Proof. Next observe that:

$$\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\lim_{\alpha\to\infty}\mathbb{E}^X_{rss}\left[(\Theta_0+\Omega_0)U'(y_0/y_{ss})\int_0^\tau \phi\exp(-(\rho+\phi)(\tau-s))v_0(s)\,ds\right] &= \\
\exp(-(\rho+\phi)\tau)\,\mathbb{E}^X_{rss}\left[(\Theta_0+\Omega_0)\int_0^\tau \phi\exp((\rho+\phi)\cdot s)U'(y_0/y_{ss})\lim_{\bar{\lambda}\to\infty}\lim_{\alpha\to\infty}\left[v_t(s)\right]ds\right] &= \\
\exp(-(\rho+\phi)\tau)\,\mathbb{E}^X_{rss}\left[(\Theta_0+\Omega_0)U'(y_0/y_{ss})\int_0^\tau \phi\exp((\rho+\phi)\cdot s)\exp(-\rho s)\,ds\right] &= \\
\exp(-(\rho+\phi)\tau)\,\mathbb{E}^X_{rss}\left[(\Theta_0+\Omega_0)U'(y_0/y_{ss})\int_0^\tau \phi\exp(\phi\cdot s)\,ds\right] &= \\
\mathbb{E}^X_{rss}\left[(\Theta_0+\Omega_0)U'(y_0/y_{ss})\right]\exp(-(\rho+\phi)\tau)(\exp(\phi\tau)-1).&
\end{aligned}$$

We make use of this expression below. □

Finally, we will obtain the result that all RSS issuance elasticities are proportional to:

$$Z(\tau) = \frac{1-\exp(-\phi\tau)}{\exp((\rho-r^*_{ss})\tau)-(1-\exp(-\phi\tau))} > 0.$$

Lemma 4. *The function* $Z(\tau)$ *is decreasing for any* $\tau > T^* \equiv \frac{1}{\phi}\log\left(\frac{\phi+(\rho-r^*_{ss})}{(\rho-r^*_{ss})}\right)$.

Proof. Thus function is of the form:

$$Z(\tau) = \frac{x(\tau)}{y(\tau) - x(\tau)},$$

and then

$$\begin{aligned}\frac{\partial Z}{\partial \tau} &= Z(\tau)\left(\frac{1}{x}\frac{dx}{d\tau} - \frac{\frac{dy}{d\tau}}{y-x} + \frac{\frac{dx}{d\tau}}{y-x}\right) = Z(\tau)\left(\frac{dx}{d\tau}\left(\frac{y-x+x}{x(y-x)}\right) - \frac{dy}{d\tau}\frac{1}{y-x}\right) \\ &= \frac{Z(\tau)}{(y-x)} \cdot \left(\frac{dx}{d\tau}\left(\frac{y}{x}\right) - \frac{dy}{d\tau}\right),\end{aligned}$$

where

$$x(\tau) \equiv 1 - \exp(-\phi\tau);\ y(\tau) \equiv \exp(\rho\tau),$$

and

$$y(\tau) - x(\tau) = \exp(\rho\tau) - 1 + \exp(-\phi\tau) > 0.$$

Thus, we have that:

$$\begin{aligned}sign\left[\frac{\partial Z}{\partial \tau}\right] &= sign\left[\frac{dx}{d\tau}\frac{1}{x} - \frac{dy}{d\tau}\frac{1}{y}\right]. \\ &= sign\left[\frac{\phi\exp(-\phi\tau)}{1-\exp(-\phi\tau)} - (\rho - r_{ss}^*)\right] \\ &= sign\left[\phi\exp(-\phi\tau) - (\rho - r_{ss}^*)(1-\exp(-\phi\tau))\right]\end{aligned}$$

$\frac{\partial Z}{\partial \tau} \leq 0$, if:

$$\frac{1}{\exp(\phi\tau) - 1} < \frac{(\rho - r_{ss}^*)}{\phi}.$$

The function $(\exp(\phi\tau) - 1)^{-1}$ is decreasing in τ and converges to infinity as $\tau \to 0$. Thus, the function is decreasing for any $\tau > \tau^*$ solving:

$$\frac{\phi + (\rho - r_{ss}^*)}{(\rho - r_{ss}^*)} < \exp(\phi T^*).$$

Which is solved by:

$$\tau^* = \frac{1}{\phi}\log\left(\frac{\phi + (\rho - r_{ss}^*)}{(\rho - r_{ss}^*)}\right) > 0.$$

□

We make use of this Lemma in the proof of the main result.

Part 1: Self-Insurance. We investigate the effect on the RSS issuances and WAM of a small drop in income from y_{ss} to y_0. We consider only a perturbation regarding income, that is:

$$y_0 = y_{ss}\left(1-\varepsilon\right)$$

Also, by assumption $\Phi\left(\cdot\right)=1$, and $\Omega_{rss}=0$.

$$\hat{v}_{rss}\left(\tau\right)=\exp\left(-\left(\rho+\phi\right)\tau\right)+U'\left(\left(1-\epsilon\right)\right)\int_0^{\tau}\phi\exp\left(-\left(\rho+\phi\right)\left(\tau-s\right)\right)v_0\left(s\right)ds.$$

Next, we evaluate the derivative of $v_{rss}\left(\tau\right)$ with respect to ε. We obtain:

$$\frac{\partial}{\partial\varepsilon}\left[\hat{v}_{rss}\left(\tau\right)\right]|_{\varepsilon=0} \;=\; \sigma\cdot\exp\left(-\left(\rho+\phi\right)\tau\right)\int_0^{\tau}\phi\exp\left(\left(\rho+\phi\right)s\right)v_0\left(s\right)ds.$$

Then, since

$$\iota_{rss}\left(\tau\right)=\frac{1}{\bar{\lambda}}\left(1-\frac{\hat{v}_{rss}\left(\tau\right)}{\hat{\psi}_{rss}\left(\tau\right)}\right)$$

we have that:

$$\frac{\partial}{\partial\varepsilon}\left[\iota_{rss}\left(\tau\right)\right]|_{\epsilon=0}=-\frac{\sigma}{\bar{\lambda}\psi_{rss}\left(\tau\right)}\cdot\exp\left(-\left(\rho+\phi\right)\tau\right)\int_0^{\tau}\phi\exp\left(\left(\rho+\phi\right)\cdot s\right)v_0\left(s\right)ds<0,$$

where $\hat{\psi}_{rss}\left(\tau\right)=\exp\left(-r_{ss}^{*}\tau\right)$. Thus, self-insurance reduces issuances. Next, we compute the semi-elasticity of issuances, with respect to the small deviation:

$$\epsilon_{rss,\epsilon}^{\tau}=\frac{1}{\iota_{rss}\left(\tau\right)}\frac{\partial}{\partial\varepsilon}\left[\iota_{rss}\left(\tau\right)\right]|_{\epsilon=0}=-\sigma\frac{\exp\left(-\left(\rho-r_{ss}^{*}+\phi\right)\tau\right)\int_0^{\tau}\phi\exp\left(\left(\rho+\phi\right)\cdot s\right)v_0\left(s\right)ds}{1-\left(\exp\left(-\left(\rho-r_{ss}^{*}+\phi\right)\tau\right)\int_0^{\tau}\phi\exp\left(\left(\rho+\phi\right)\cdot s\right)v_0\left(s\right)ds\right)}.$$

Thus, the elasticity becomes:

$$\begin{aligned}\lim_{\bar{\lambda}\to\infty}\lim_{\alpha\to\infty}\epsilon^{\tau}_{rss,\varepsilon} &= -\sigma\frac{\exp\left(-\left(\rho-r^{*}_{ss}+\phi\right)\tau\right)\left(\exp\left(\phi\tau\right)-1\right)}{1-\exp\left(-\left(\rho-r^{*}_{ss}+\phi\right)\tau\right)\left(\exp\left(\phi\tau\right)-1\right)}\\ &= -\sigma\frac{\exp\left(\phi\tau\right)-1}{\exp\left(\left(\rho-r^{*}_{ss}+\phi\right)\tau\right)-\left(\exp\left(\phi\tau\right)-1\right)}\\ &= -\sigma\frac{1-\exp\left(-\phi\tau\right)}{\exp\left(\left(\rho-r^{*}_{ss}\right)\tau\right)+\exp\left(-\phi\tau\right)-1}\\ &= -\sigma\cdot Z\left(\tau\right).\end{aligned}$$

Thus, since $Z\left(\tau\right)$ is decreasing for $\tau > \tau^{*}$, the limit elasticity $\lim_{\bar{\lambda}\to\infty}\lim_{\alpha\to\infty}\epsilon^{\tau}_{rss,\epsilon}$ is increasing in τ. Thus, if the government can only issue at maturities $\tau > \tau^{*}$, self-insurance depresses issuances and increases the WAM.

Part 2: Credit-Risk. We now consider the limit where there are no shocks, and only the option to default is allowed. Let the Government be risk neutral, $\sigma = 0$. As $\bar{\lambda}\to\infty$, we get $\Omega_0 \approx 0$. Then,

$$\hat{v}_{rss}\left(\tau\right) = \exp\left(-\left(\rho+\phi\right)\tau\right) + \phi\left(\Theta_0+\varepsilon\right)\int_0^{\tau}\exp\left(-\left(\rho+\phi\right)\left(\tau-s\right)\right)v_0\left(s\right)ds.$$

Likewise, the price is:

$$\hat{\psi}_{rss}\left(\tau\right) = \exp\left(-\left(r^{*}_{ss}+\phi\right)\tau\right) + \phi\left(\Theta_0+\varepsilon\right)\int_0^{\tau}\exp\left(-\left(r^{*}_{ss}+\phi\right)\left(\tau-s\right)\right)\psi_0\left(s\right)ds.$$

Then, as show before, we have that:

$$\lim_{\bar{\lambda}\to\infty} v_t\left(s\right) = \exp\left(-\rho s\right)$$

and

$$\psi_t\left(s\right) = \exp\left(-r^{*}_{ss}s\right),$$

we obtain:

$$\hat{v}_{rss}\left(\tau\right) = \exp\left(-\left(\rho+\phi\right)\tau\right)\left(1+\left(\Theta_0+\varepsilon\right)\left(\exp\left(\phi\tau\right)-1\right)\right).$$

Thus:

$$\frac{\partial}{\partial\varepsilon}\left[\hat{v}_{rss}\left(\tau\right)\right]|_{\varepsilon=0} = \exp\left(-\left(\rho+\phi\right)\tau\right)\left(\exp\left(\phi\tau\right)-1\right)$$

and by analogy:

$$\frac{\partial}{\partial \varepsilon}\left[\hat{\psi}_{rss}(\tau)\right]|_{\varepsilon=0} = \exp\left(-\left(r_{ss}^{*}+\phi\right)\tau\right)\left(\exp\left(\phi\tau\right)-1\right)$$

Then:

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon}\left[\iota_{rss}(\tau)\right]|_{\varepsilon=0} &= -\frac{1}{\bar{\lambda}}\left(\frac{\hat{v}_{rss}(\tau)}{\hat{\psi}_{rss}(\tau)}\right)\left(\frac{\frac{\partial}{\partial \varepsilon}\left[\hat{v}_{rss}(\tau)\right]}{\hat{v}_{rss}(\tau)}-\frac{\frac{\partial}{\partial \varepsilon}\left[\hat{\psi}_{rss}(\tau)\right]}{\hat{\psi}_{rss}(\tau)}\right)|_{\epsilon=0} \\
&= -\frac{1}{\bar{\lambda}}\left(\frac{\exp\left(-\left(\rho+\phi\right)\tau\right)}{\exp\left(-\left(r_{ss}^{*}+\phi\right)\tau\right)}\right)\left(\frac{\frac{\partial}{\partial \varepsilon}\left[\hat{v}_{rss}(\tau)\right]}{\hat{v}_{rss}(\tau)}-\frac{\frac{\partial}{\partial \varepsilon}\left[\hat{\psi}_{rss}(\tau)\right]}{\hat{\psi}_{rss}(\tau)}\right)|_{\epsilon=0} \\
&= 0,
\end{aligned}$$

where we have used that:

$$\frac{\frac{\partial}{\partial \varepsilon}\left[\hat{v}_{rss}(\tau)\right]}{\hat{v}_{rss}(\tau)}|_{\varepsilon=0} = \frac{\exp\left(-\left(\rho+\phi\right)\tau\right)\left(\exp\left(\phi\tau\right)-1\right)}{\exp\left(-\left(\rho+\phi\right)\tau\right)\left(1+\left(\Theta_0\right)\left(\exp\left(\phi\tau\right)-1\right)\right)} = \frac{\left(\exp\left(\phi\tau\right)-1\right)}{1+\Theta_{rss}\left(\exp\left(\phi\tau\right)-1\right)}.$$

and

$$\frac{\frac{\partial}{\partial \varepsilon}\left[\hat{\psi}_{rss}(\tau)\right]}{\hat{\psi}_{rss}(\tau)}|_{\varepsilon=0} = \frac{\left(\exp\left(\phi\tau\right)-1\right)}{1+\Theta_{rss}\left(\exp\left(\phi\tau\right)-1\right)}.$$

Thus, issuances are unresponsive to changes in the default probability. Thus, the WAM does not change with credit risk.

Part 3: Revenue Echo. We now consider a shock to the revenue echo effect, while maintaining the assumption that $\sigma = 0$ and $\bar{\lambda} \to \infty$. As before we have that:

$$\begin{aligned}
\hat{v}_{rss}(\tau) &= \exp\left(-\left(\rho+\phi\right)\tau\right)+\phi\left(\Theta_0+\Omega_0+\varepsilon\right)\int_0^{\tau}\exp\left(-\left(\rho+\phi\right)\left(\tau-s\right)\right)v_0(s)\,ds, \\
&= \exp\left(-\left(\rho+\phi\right)\tau\right)\left(1+\left(\Theta_0+\Omega_0+\varepsilon\right)\left(\exp\left(\phi\tau\right)-1\right)\right)
\end{aligned}$$

with $\Omega_0 \approx 0$. Thus, the derivative is:

$$\lim_{\bar{\lambda}\to\infty}\frac{\partial}{\partial \varepsilon}\left[\hat{v}_{rss}(\tau)\right]|_{\varepsilon=0} = \exp\left(-\left(\rho+\phi\right)\tau\right)\left(\exp\left(\phi\tau\right)-1\right).$$

Bond prices are

$$\begin{aligned}\hat{\psi}_{rss}(\tau) &= \exp(-(r_{ss}^{*}+\phi)\tau)+\phi\Theta_0\int_0^{\tau}\exp(-(r_{ss}^{*}+\phi)(\tau-s))\psi_0(s)\,ds \\ &= \exp(-(r_{ss}^{*}+\phi)\tau)(1+\Theta_0(\exp(\phi\tau)-1))\end{aligned}$$

and

$$\frac{\partial}{\partial\varepsilon}\left[\hat{\psi}_{rss}(\tau)\right]|_{\varepsilon=0}=0.$$

Issuances are

$$\begin{aligned}\iota_{rss}(\tau) &= \frac{1}{\bar{\lambda}}\left[1-\frac{\exp(-(\rho+\phi)\tau)(1+(\Theta_0+\Omega_0+\varepsilon)(\exp(\phi\tau)-1))}{\exp(-(r_{ss}^{*}+\phi)\tau)(1+\Theta_0(\exp(\phi\tau)-1))}\right] \\ &= \frac{1}{\bar{\lambda}}\left[1-\exp(-(\rho-r_{ss}^{*})\tau)\right],\end{aligned}$$

where we have incorporated the fact that $\Omega_0\approx 0$, $\varepsilon\approx 0$. Then, we have that:

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\left[\iota_{rss}(\tau)\right]|_{\epsilon=0} &= -\frac{1}{\bar{\lambda}}\left(\frac{\frac{\partial}{\partial\varepsilon}\left[\hat{v}_{rss}(\tau)\right]}{\hat{\psi}_{rss}(\tau)}\right)|_{\epsilon=0}. \\ &= -\frac{1}{\bar{\lambda}}\frac{\exp(-(\rho+\phi)\tau)(\exp(\phi\tau)-1)}{\exp(-(r_{ss}^{*}+\phi)\tau)(1+\Theta_0(\exp(\phi\tau)-1))} \\ &= -\frac{1}{\bar{\lambda}}\frac{\exp(-(\rho-r_{ss}^{*})\tau)(\exp(\phi\tau)-1)}{(1+\Theta_0(\exp(\phi\tau)-1))}\end{aligned}$$

Then, following previous steps, the semi-elasticity is

$$\lim_{\bar{\lambda}\to\infty}\epsilon_{rss,\varepsilon}^{\tau} = -\frac{\exp(-(\rho-r_{ss}^{*})\tau)(\exp(\phi\tau)-1)}{[1-\exp(-(\rho-r_{ss}^{*})\tau)](1+\Theta_0(\exp(\phi\tau)-1))}<0$$

If $\Theta_0\approx 1$, this simplifies to

$$\lim_{\bar{\lambda}\to\infty}\epsilon_{rss,\varepsilon}^{\tau}=-\frac{(1-\exp(-\phi\tau))}{[\exp((\rho-r_{ss}^{*})\tau)-1]}.$$

We compute the slope as .

$$\lim_{\bar{\lambda}\to\infty}\frac{\partial\epsilon_{rss,\varepsilon}^{\tau}}{\partial\tau}=\epsilon_{rss,\varepsilon}^{\tau}\left[\frac{\phi\exp(-\phi\tau)}{(1-\exp(-\phi\tau))}-\frac{(\rho-r_{ss}^{*})\exp((\rho-r_{ss}^{*})\tau)}{[\exp((\rho-r_{ss}^{*})\tau)-1]}\right].$$

This is positive if

$$\frac{\phi}{(\exp(\phi\tau)-1)} < \frac{(\rho - r^*_{ss})}{[1-\exp(-(\rho - r^*_{ss})\tau)]},$$

which holds for any τlarger than τ^{**} defined as

$$\frac{\phi}{(\exp(\phi\tau^{**})-1)} = \frac{(\rho - r^*_{ss})}{[1-\exp(-(\rho - r^*_{ss})\tau^{**})]}.$$

It is trivial to check that $\tau^{**} = 0.$ Hence the semi-elasticity is increasing and the WAM increases.