Monetary Policy and Sovereign Debt Sustainability*

Samuel Hurtado — Banco de España
Galo Nuño — Banco de España
Carlos Thomas — Banco de España

First version: March 2015
This version: February 2022

Abstract

We analyze the consequences of monetary policy for sovereign debt sustainability and welfare, in a model of a small open economy where the government issues long-term nominal debt without commitment not to default on it or erode its real value through (costly) inflation. Inflation is a form of *partial default*, one that is more state-contingent than outright default. This reduces the government's incentives to default outright and hence enlarges the repayment region, compared to a regime in which debt cannot be inflated away. Moreover, inflation delivers sizable welfare gains in situations of sovereign debt stress, in which its benefits as a debt-stabilizing tool are larger. Over the longer run, however, the welfare gains from inflation are more modest, because the inflationary bias leads the government to create inflation also in situations in which it is less useful for debt-stabilization purposes.

Keywords: monetary and fiscal policy, discretion, fundamental sovereign default, inflationary bias, continuous time.

JEL codes: E5, E62, F34.

*The views expressed in this paper are those of the authors and do not necessarily represent the views of Banco de España or the Eurosystem. This manuscript was previously circulated as "Monetary Policy and Sovereign Debt Vulnerability". The authors are very grateful to Fernando Álvarez, Manuel Amador, Óscar Arce, Cristina Arellano, Javier Bianchi, Antoine Camous, Giancarlo Corsetti, Jose A. Cuenca, Luca Dedola, Antonia Diaz, Aitor Erce, Maurizio Falcone, Esther Gordo, Jonathan Heathcote, Alfredo Ibañez, Fabio Kanczuk, Peter Karadi, Matthias Kredler, Xiaowen Lei, Albert Marcet, Luis Molina, Benjamin Moll, Juan P. Nicolini, Ricardo Reis, Juan P. Rincón-Zapatero, Pedro Teles, conference participants at the SED Annual Meeting, REDg Workshop, Oxford-FRB of New York Monetary Economics Conference, Barcelona GSE Summer Forum, Banco do Brasil Annual Inflation Targeting Seminar, Theories and Methods in Macroeconomics Conference, European Winter Meetings of the Econometric Society, and seminar participants at Université Paris-Dauphine, Universidad Carlos III, FRB of Minneapolis, ECB, Banca d'Italia, BBVA, Banco de España, European Commission, Universidad de Vigo, and University of Porto for helpful comments and suggestions. All remaining errors are ours.

1 Introduction

The economic response to the Covid-19 pandemic has increased debt-to-GDP ratios near or above record levels for many advanced and emerging economies. This has led many observers to fret about how these high debt ratios will be sustained in the future. The use of inflation as a tool to erode the real value of debt has often been mentioned.[1] While inflation can effectively alleviate the real burden of debt ex-post, it entails some costs for citizens that should also be taken into account. Furthermore, an inflationary policy will raise bond yields today if anticipated by bond markets, thus increasing the costs of debt servicing and undermining the very aim of the policy. This paper tackles these issues, analyzing how the ability to inflate debt away affects sovereign debt sustainability and welfare outcomes.

We address the above question by studying the implications of inflationary policy when the government cannot commit not to default explicitly on its debt, but also not to reduce its real value through inflation. We do so in the context of a standard quantitative model of optimal sovereign default *à la* Aguiar and Gopinath (2006) and Arellano (2008). As in Hatchondo and Martínez (2009) and Chatterjee and Eyigungor (2012), we consider a small open economy populated by risk-averse households in which a benevolent government issues long-term non-contingent bonds to foreign investors. The government may default on its debt at any time, but at the expense of temporary exclusion from capital markets (thus losing the ability to smooth consumption) and a drop in output during the exclusion period. In order to introduce a role for monetary policy, we depart from the standard literature by assuming that debt is non-contingent in *nominal* (domestic-currency) terms, such that inflation erodes its real value; inflation however entails welfare costs, thus creating a meaningful trade-off for monetary policy. The government chooses optimal fiscal and monetary policy under discretion, i.e. without commitment on the future path of primary deficits or inflation.

We calibrate our model to Brazilian data. We choose Brazil as our case study for two reasons. First, it has a relatively high share of sovereign debt denominated in domestic currency, a prerequisite for debt inflation policies to be effective. Second, its sovereign debt crisis in 2002-03 provides a good example of a situation in which high inflation and fears of outright default go hand in hand, as they do in our model when the economy lies close to the default frontier. Our calibration targets include not only sample moments, but also targets related to the 2002-03 episode, such as the increase in inflation, the rise in sovereign spreads, and the contribution to the latter rise of the inflation vs the default premium.

Our analysis first highlights the properties of the optimal default and inflation policies. As in the standard literature, the default frontier is upward sloping in the debt-income space: sovereign default is optimal when debt is high and/or income is low. As regards inflation, we show analytically that it depends on two factors. First, the real value of debt: as long as there is debt outstanding, the government has an incentive to inflate it away. This is a reflection of the "inflationary bias" that arises under discretion in the presence of nominal government debt.[2]

[1] See, for instance, Dixon (2020) or Kelton and Chancellor (2020).

[2] See e.g. Díaz-Giménez et al. (2008), Martín (2009), Niemann (2011), and Niemann, Pichler and Sorger

Second, inflation increases with the welfare gain from a marginal reduction in the real value of debt. We show numerically that this gain is roughly constant except for debt values close to the default frontier. These two forces amount to an inflation policy that increases essentially linearly with debt and then stabilizes at high values near the default frontier.

We then analyze the impact of optimal inflation policy on sovereign debt sustainability, understood as the government's incentives to default outright on its debt as opposed to (partially) defaulting on it through inflation. We do so by comparing our inflationary equilibrium with a counterfactual equilibrium in which inflation is zero at all times ("no-inflation regime"), which can be interpreted as a scenario in which the government issues foreign-currency debt, or joins an anti-inflationary monetary union, thus effectively renouncing the use of inflation. We show that inflation *enlarges* the repayment region by shifting out the default frontier: at any income level, the debt level above which the government prefers default to repayment is higher. The reason is the following. Inflation is a form of *partial default*, as it allows to reduce the real debt burden, only it is a more continuous and state-contingent form of default than outright default. This state-contingency allows the government to smooth consumption in the face of aggregate income shocks, the more so the higher the debt burden. Thus, the possibility of using inflation reduces the incentive to use outright default in situations of high debt and/or low output.

We then turn to the welfare implications of discretionary inflation, which pivot around the following basic trade-off. On the one hand, and as said before, partial default through inflation allows the government to respond to aggregate shocks in a more state-contingent manner than outright default, a comparatively blunt tool. This results in a higher present-discounted value of future expected consumption utility flows. On the other hand, the inflationary bias incurred by the government entails a welfare cost. We find that, in a region of the state space close to the default frontier, welfare is *higher* in the *inflationary* regime, i.e. the benefits from state-contingent inflation dominate the cost from the inflationary bias. The reason is the following. In present-discounted terms, the cost of inflation is relatively stable across debt levels, as households anticipate that inflation will be relatively high most of the time. By contrast, the benefits from state-contingent inflation in terms of higher present-discounted consumption utility flows grow with debt, because the higher the debt the more effective inflation is at inflating it away. As a result, it is for debt levels close to the default threshold that inflation delivers larger welfare gains. Quantitatively, the welfare gains from inflation *vis-à-vis* the no-inflation regime –conditional on states in which the government honors its debt in both regimes– can be as high as 0.12% of permanent consumption.

We next employ our model to analyze the Brazilian debt crisis of 2002-03. In 2002 Brazil experienced a surge in sovereign spreads, amid fears of a sovereign default. This was followed by a substantial rise in inflation which, together with a primary deficit surplus, reduced its debt-to-GDP ratio. As mentioned before, our model is calibrated (partly) to reproduce the evolution of inflation, the spread on domestic-currency sovereign debt and the contribution of the default vs inflation premium to such spreads during this episode, which the model does

(2013).

reasonably well despite its parsimoniousness. We then show that, in a counterfactual scenario without inflation, Brazil would have *defaulted* outright on its debt. As a result, welfare would have been as much as 0.26% lower than in the inflationary scenario.

Finally, we show that *average* welfare gains from inflation are more modest than the conditional gains close to default. As explained before, away from the default frontier inflation can be welfare-reducing, because it is less useful as a debt-stabilizing tool but the inflationary bias continues to entail sizable costs. Since in the ergodic debt distribution the economy does spend some time away from the default frontier, when integrating conditional welfare gains across the state space the average gains from inflation turn out to be smaller than in situations of sovereign debt stress. We also find that average welfare gains from inflation are larger the higher the standard deviation of aggregate shocks, confirming the fact that inflation is valuable because of its role as a state-contingent shock absorber –the more so the larger the amplitude of the shocks hitting the economy.

In sum, our analysis suggests that discretionary inflation may deliver material welfare gains in situations of sovereign debt stress, such as the one experienced by Brazil in 2002-03. However, over the longer run, such gains are likely to be smaller, because away from default inflation is less useful as a partial default tool whereas the cost from the inflationary bias continues to take its toll on welfare.

Related literature. Our paper is related to the literature on quantitative models of optimal sovereign default in small open economies initiated by Aguiar and Gopinath (2006) and Arellano (2008), who in turn built on the seminal qualitative framework of Eaton and Gersovitz (1981).[3] Our framework is closest to the models with long-term bonds developed by Hatchondo and Martínez (2009) and Chatterjee and Eyigungor (2012).[4] We contribute to this literature by proposing a model with nominal debt and costly inflation and analyzing the implications of optimal discretionary monetary policy for sovereign debt sustainability and welfare. In this regard, our analysis is related to Sunder-Plassmann (2020) and Röttger (2019), who introduce optimal default *à la* Arellano (2008) into closed-economy frameworks with monetary frictions *à la* Díaz-Giménez et al. (2008) and Martin (2009). Apart from differences in modelling and in the relevant channels,[5] our papers differ largely in focus. Sunder-Plassmann (2020) studies how the denomination of sovereign debt (nominal vs. real) affects the government's incentives to inflate or default on its debt over the long run. Röttger (2019) focuses on how the ability to default changes the conduct of monetary and fiscal policy in the short and long run. By contrast, we analyze how a government's ability to inflate away its nominal, local-currency-denominated

[3]For an in-depth review of the literature on quantitative models of sovereign default and more generally of sovereign debt crises, see Aguiar et al. (2016).

[4]In fact, our model is essentially a continuous-time version of Chatterjee and Eyigungor (2012) with nominal debt and inflation costs. We assume continuous time because it offers considerable computational advantages, as discussed later on.

[5]In Sunder-Plassmann's (2020) and Röttger's (2020) closed-economy setups with monetary frictions, inflating away the debt (or reducing it through outright default) allows the government to reduce future distortionary taxation, including the inflation tax on consumption goods purchased with cash. In our cashless, open-economy framework, by contrast, inflation produces a redistribution from foreign investors to the domestic economy, which is closer in spirit to the channel through which default favors welfare in the standard open-economy model of sovereign default (e.g. Arellano, 2008; Aguiar and Gopinath, 2006).

debt affects not only its sustainability but also social welfare.[6]

Our analysis stresses the role of inflation as a partial default tool. In this regard, it is related to Arellano, Mateos-Planas and Rios-Rull (2019), who analyze a model of partial outright default in which the government endogenously chooses the fraction of debt to renege on, at a cost that depends on this fraction. They show that their model can replicate the main stylized facts related to the intensity and duration of sovereign defaults in the data.

Our paper is more loosely related to the literature that analyzes, in the context of sovereign debt models with multiple equilibria in the tradition of Calvo (1988) and Cole and Kehoe (2000), to what extent monetary policy can eliminate the possibility of self-fulfilling debt crises. Examples of this research are Aguiar et al. (2013, 2015), Reis (2013), Da Rocha, Giménez and Lores (2013), Araujo, León and Santos (2013), Corsetti and Dedola (2016), Camous and Cooper (2019), and Bacchetta, Perazzi and van Wincoop (2018). Unlike these contributions, we do not consider self-fulfilling debt crises, focusing instead on sovereign default due only to bad fundamentals.[7] Conversely, none of the above papers analyze the consequences of monetary policy for debt sustainability and welfare in a quantitative, fully dynamic economy with recurrent aggregate shocks and optimal fundamental default, as we do.[8] A key insight from our analysis is that discretionary inflation not only improves sovereign debt sustainability –by pushing out the debt-income frontier beyond which the country suffers (fundamental) default–, but also improves welfare when the economy is sufficiently close to default. While not directly comparable, this stands somewhat in contrast with one of the key lessons from the above literature, according to which the use of discretionary inflation policy generally backfires by not avoiding self-fulfilling debt crisis and yet causing welfare losses (see e.g. Calvo, 1988, or more recently Corsetti and Dedola, 2016).[9]

In modelling the choice of inflation without commitment as a trade-off between the reduction in the real debt burden *vis-à-vis* foreign investors and the utility costs of inflation, our model bears some resemblance with Aguiar et al. (2013). Apart from this aspect, both papers differ notably in modelling, focus and findings. Aguiar et al. (2013) study the effects of the utility costs of inflation –which the authors refer to as the government's "inflation credibility"– on the potential for self-fulfilling debt crises, in a qualitative model without fundamental un-

[6]Also related is the work of Du and Schreger (2017), who analyze how the denomination of *corporate* debt determines the sovereign's incentive to inflate or default on its (local-currency-denominated) debt, in an Aguiar-Gopinath-Arellano economy extended to allow for firms that face borrowing constraints and a currency mismatch between revenues and liabilities.

[7]By contrast, the papers above consider only default due to self-fulfilling expectations. An exception is Corsetti and Dedola's (2016) qualitative framework, where default can also be due to weak fundamentals.

[8]Many of the above contributions are qualitative, working in environments with two periods or two-period-lived agents (e.g. Corsetti and Dedola, 2016; Camous and Cooper, 2019) or without fundamental uncertainty (e.g. Aguiar et al., 2013, 2015). Bacchetta et al. (2018) propose a dynamic framework where fundamental uncertainty is restricted to the value of primary deficit at some future date. Da Rocha et al. (2013) and Araujo et al. (2013) study fully dynamic, stochastic environments with self-fulfilling debt crises; however, they do not discuss the role of discretionary inflation for debt sustainability and welfare that is central to our analysis.

[9]Corsetti and Dedola (2016) qualify the Calvo (1988) result by showing that, if inflation costs are convex, then multiplicity disappears and the equilibrium is unique. Camous and Cooper (2019), and Bacchetta, Perazzi and van Wincoop (2018) show that optimal monetary policy under *commitment* can be successful at eliminating self-fulfilling debt crisis.

certainty where failure by investors to roll over the debt may lead the government to choose outright default over full principal repayment. Aguiar et al. (2013) find that, if inflation costs are below a certain threshold, inflationary policy (i) makes the economy *more vulnerable* by reducing the debt threshold above which the economy is exposed to self-fulfilling crises and (ii) achieves strictly *lower* welfare for any debt level, *vis-à-vis* a scenario with foreign currency debt (analogous to our 'no-inflation' regime). By contrast, we evaluate the consequences of discretionary inflation for fundamentally-driven default in a quantitative stochastic framework. We find that the inflationary regime (i) *improves* debt sustainability by shifting out the default frontier, and (ii) achieves *higher* welfare when the economy is close to the default frontier.

While our paper focuses on the role of inflation as a debt-management tool in economies with debt issued in local currency, three recent papers analyze the interactions between default and monetary policy in economies with foreign-currency debt. Na, Schmitt-Grohé, Uribe and Yue (2018) study a sovereign default model with downward nominal wage rigidity and show that it can account for the joint occurrence of large nominal devaluations and defaults. Arellano, Mihalache and Bai (2019) develop a New Keynesian model with sovereign default risk to study the interactions between monetary policy and default. Bianchi and Mondragon (2019) study a sovereign default model with self-fulfilling rollover crises and downward nominal wage rigidity. They show that the inability to use monetary policy for macroeconomic stabilization leaves a government more vulnerable to a rollover crisis as lenders anticipate that the government would face a severe recession in the event of a liquidity crisis, and are therefore more prone to run on government bonds.

Finally, we make a technical contribution by laying out a quantitative optimal sovereign default model in continuous time and introducing a new numerical method to compute the equilibrium. As discussed in Achdou et al. (2021), the computational burden is reduced in continuous-time dynamic programming compared to discrete-time methods. A number of subsequent works have made further progress in analyzing continuous-time quantitative models of optimal sovereign default. Tourre (2017) considers a model that allows for semi-closed form solutions to disentangle which model features influence credit spreads, expected returns and cross-country correlations. Bornstein (2020) compares, in a continuous-time version of Arellano (2008), the equilibrium dynamics and computing times to those of discrete-time models, finding that continuous-time techniques are faster. Rebelo, Wang and Yang (2021) analyze a model of sovereign default in which countries vary in their level of financial development using a continuous-time approach.

2 Model

We consider a continuous-time model of a small open economy.

2.1 Output, price level and sovereign debt

There is a single, freely traded consumption good which has an international price normalized to one. The economy is endowed with y_t units of the good each period (real GDP). The evolution of $z_t = \log(y_t)$ is given by a bounded Ornstein–Uhlenbeck process (the continuous-time counterpart of the AR(1))

$$dz_t = -\mu z_t dt + \sigma dW_t, \tag{1}$$

The local currency price relative to the World price at time t is denoted P_t. It evolves according to

$$dP_t = \pi_t P_t dt, \tag{2}$$

where π_t is the instantaneous inflation rate. Notice that, under our assumption of a single freely-traded good, the law of one price holds and the price level must be the same when converted into the same currency. Therefore, the nominal exchange rate coincides with the domestic price level.[10]

The government trades a nominal non-contingent bond with risk-neutral competitive foreign investors. Let B_t denote the outstanding stock of nominal government bonds; assuming that each bond has a nominal value of one unit of domestic currency, B_t also represents the total nominal value of outstanding debt. We assume that outstanding debt is amortized at rate $\lambda > 0$ per unit of time. The nominal value of outstanding debt thus evolves as follows,

$$dB_t = B_t^{new} dt - \lambda B_t dt,$$

where B_t^{new} is the flow of new debt issued at time t. Each bond pays a proportional coupon δ per unit of time.[11] The nominal market price of government bonds at time t is Q_t. Also, the government incurs a nominal primary deficit $P_t (c_t - y_t)$, where c_t is aggregate consumption.[12] The government's flow of funds constraint is then

$$Q_t B_t^{new} = (\lambda + \delta) B_t + P_t (c_t - y_t).$$

That is, the proceeds from the issuance of new bonds must cover amortization and coupon payments plus the primary deficit. Combining the last two equations, we obtain the following

[10]Letting P_t^* and E_t denote the World price level and the nominal exchange rate, respectively, it holds that$P_t = P_t^* E_t$. Given our normalization $P_t^* = 1$, we have $P_t = E_t$.

[11]Our modeling of long-term nominal debt, with bonds amortized at a constant rate and with fixed coupon rate, is similar to the nominal perpetual bonds with geometrically decaying coupons introduced by Woodford (2001) in a discrete-time macroeconomic framework. In fact, the latter bonds can be interpreted as a particular case of the bonds considered here, with the amortization and coupon rate adding up to one ($\lambda + \delta = 1$). See also Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012) for recent uses of similar modelling devices for long-term (real) bonds in discrete-time open economy setups.

[12]As in Arellano (2008), we assume that the government rebates back to households all the net proceedings from its international credit operations (i.e. its primary deficit) in a lump-sum fashion. Denoting by $\tilde{T}_t$ the primary deficit, we thus have $P_t c_t = P_t y_t + \tilde{T}_t$. This implies $\tilde{T}_t = P_t (c_t - y_t)$.

dynamics for nominal debt outstanding,

$$dB_t = \left[\frac{(\lambda + \delta) B_t + P_t (c_t - y_t)}{Q_t} - \lambda B_t \right] dt. \tag{3}$$

We define real debt in face value terms as $b_t \equiv B_t / P_t$. Its dynamics are given by

$$db_t = [\frac{(\lambda + \delta) b_t + c_t - y_t}{Q_t} - (\lambda + \pi_t) b_t] dt. \tag{4}$$

Equation (4) encapsulates two key effects of inflation on debt accumulation. First, contemporaneous inflation π_t erodes the real value of nominal debt through the classical debt inflation channel. Second, as we will see later on, the price of the long-term nominal bond, Q_t, declines with expectations of future inflation during the life of the bond, thus making it more expensive for the government to issue new bonds.

2.2 Preferences

The representative household has preferences over paths for consumption and domestic inflation given by

$$U_0 \equiv \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_t) - x(\pi_t, y_t) dt \right]. \tag{5}$$

Instantaneous utility takes the form

$$u(c) = \begin{cases} \log(c), & \text{if } \gamma = 1 \\ \frac{c^{1-\gamma}-1}{1-\gamma}, & \text{f } \gamma \neq 1 \end{cases}, \quad x(\pi, y) = \frac{\psi(y)}{2} \pi^2, \ \psi(y) = \psi y^{\zeta}. \tag{6}$$

where $\psi > 0$. The functional form for the utility costs of inflation, $\psi(y) \pi^2/2$, can be justified on the grounds of costly price adjustment by firms. In particular, in Appendix A we lay out an economy where firms are explicitly modelled, and where a subset of them are price-setters but incur a standard quadratic cost of price adjustment *à la* Rotemberg (1982). As we show there, social welfare in such an economy can be expressed as in equations (5) and (6) with $\gamma = 1$, and the relevant equilibrium conditions are identical to those in the simple model described here.[13]

2.3 Fiscal and monetary policy

The government chooses fiscal policy at each point in time along two dimensions: it sets optimally consumption c_t, and it chooses whether to continue honoring debt repayments or else to default on its debt holdings. In addition, the government implements monetary policy by choosing the inflation rate π_t at each point in time. Before analyzing the government's problem, we present first the sovereign default scenario.

[13]To be precise, the quadratic utility cost $\psi(y) \pi^2/2$ is an approximation to the exact utility cost of inflation in the model of Appendix A.

2.3.1 The default scenario

The government may default on its debt. Following most of the literature on quantitative sovereign default models (e.g. Aguiar and Gopinath, 2006; Arellano, 2008), we assume that a default entails two types of costs. First, the government is excluded from international capital markets temporarily. The duration of this exclusion period, τ, is random and follows an exponential distribution with average duration $1/\chi$. Second, during the exclusion period the country's output endowment declines. Suppose the government defaults at an arbitrary debt level b. Then during the exclusion period the country's output endowment is given by $y_t^{def} = y_t - \epsilon(y_t)$, with $\epsilon(\cdot)$ being the output loss. This specification of output loss is similar to the one in Arellano (2008) and Chatterjee and Eyigungor (2012). During the exclusion phase, households simply consume the output endowment, $c_t = y_t^{def}$.

The main benefit of defaulting is of course the possibility of reducing the debt burden. During the exclusion period, which may be interpreted as a renegotiation process between the government and the investors, the latter receive no repayments. Let $\tilde{t}$ denote the time of the most recent default. We assume that at the end of the exclusion period, i.e. at time $\tilde{t}+\tau$, both parties reach an agreement by which investors recover a fraction θ of the nominal value of outstanding bonds at the time of default, for some parameter $\theta > 0$.[14] Importantly, it allows us to keep real debt in face value as the relevant state variables. To see this, notice that upon regaining access to capital markets, real debt is

$$b_{\tilde{t}+\tau} = \theta B_{\tilde{t}} \frac{1}{P_{\tilde{t}+\tau}} = \theta b_{\tilde{t}} \frac{P_{\tilde{t}}}{P_{\tilde{t}+\tau}} = \theta b_{\tilde{t}} e^{-\int_{\tilde{t}}^{\tilde{t}+\tau} \pi_s ds} = \theta b_{\tilde{t}+\tau}, \tag{7}$$

where $b_{\tilde{t}+\tau} = B_{\tilde{t}+\tau}/P_{\tilde{t}+\tau}$ is real debt at the time of reentry. Taking derivatives in equation (7) we obtain the law of motion of real debt while in exclusion:

$$db_t = -\pi_t b_t dt. \tag{8}$$

2.3.2 The general problem

At every point in time the government decides optimally whether to default or not, in addition to choosing consumption and inflation. If the government decides to default it chooses also the default intensity. Following a default, and once the government regains access to capital markets, it starts accumulating debt and is confronted again with the choice of defaulting. This is a sequence of *optimal stopping* problems, as one of the policy instruments is a sequence of stopping times. We denote by T the *time to default*. The latter is a stopping time, defined as the smallest time t' such that the government decides to default.[15] The government maximizes social welfare under discretion. When doing so, it takes as given the bond price schedule $Q(b,z)$, which determines how investors price government bonds in each state and which is characterized

[14] See Benjamin and Wright (2009) and Yue (2010) for studies that endogenize the recovery rate upon default, in models with explicit renegotiation between the government and its creditors.

[15] Therefore, the time of default in absolute time is $\tilde{t} = t + T$.

below. The government thus maximizes households' utility (5) subject to the laws of motion of income (1) and debt (4). The value function of the government during repayment spells is defined as

$$V(b,z) = \max_{T, d_T, \{c_t, \pi_t\}_{t\in[0,T)}} \mathbb{E}\left\{ \int_0^T e^{-\rho t} \left(u(c_t) - x\left(\pi_t, e^{z_t}\right)\right) dt + e^{-\rho T} V_{def}(b_T, z_T) \mid b_0 = b,\ z_0 = z \right\}. \tag{9}$$

The value function must satisfy a so-called "HJB Variational Inequality" (Øksendal, 1995; Pham, 2009):

$$0 = \max\left\{ V_{def}(b,z) - V(b,z), \max_{c,\pi} u(c) - x(\pi, e^z) + s(b,z,c,\pi)\frac{\partial V}{\partial b} - \mu z \frac{\partial V}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial z^2} - \rho V(b,z) \right\}, \tag{10}$$

where

$$s(b,z,c,\pi) = \left[\frac{\lambda+\delta}{Q(b,z)} - (\lambda+\pi)\right] b + \frac{c-e^z}{Q(b,z)}, \tag{11}$$

is the *drift* of the state variable b_t (see equation 4) The first order conditions of this problem imply the following policy functions for consumption and inflation,

$$u'(c(b,y)) = -\frac{\partial V}{\partial b}\frac{1}{Q(b,z)}, \tag{12}$$

$$\pi(b,z) = -\frac{1}{\psi(e^z)} b \frac{\partial V}{\partial b}. \tag{13}$$

Therefore, the optimal consumption increases with bond prices and decreases with the slope of the value function (in absolute value). The intuition is straightforward. Higher bond prices make it cheaper for the government to finance primary deficits. Likewise, a steeper value function makes it more costly to increase the debt burden by incurring in primary deficits. As regards optimal inflation, the latter increases both with debt and the slope (in absolute value) of the value function. Intuitively, the higher the debt level the larger the reduction in the debt burden that can be achieved through a marginal increase in inflation. Similarly, a steeper value function increases the incentive to use inflation so as to reduce the debt burden.

The optimal default policy $d(b,z)$ may take only two values: 1 (default) or 0 (no default),

$$d(b,z) = \begin{cases} 1, & \text{if } V_{def}(b,z) > V(b,z), \\ 0, & \text{if } V_{def}(b,z) \le V(b,z), \end{cases} \tag{14}$$

that is, default only happens when the value of defaulting is higher than that of repayment.

During the exclusion phase the government may still choose optimally the level of inflation. The value of defaulting is given by

$$V_{def}(b,z) = \max_{\{\pi_t\}_{t\in[0,T)}} \mathbb{E}\left\{ \int_0^\tau e^{-\rho t} \left(u\left(e^{z_t} - \epsilon\left(e^{z_t}\right)\right) - x\left(\pi_t, e^{z_t} - \epsilon\left(e^{z_t}\right)\right)\right) dt + e^{-\rho\tau} V\left(\theta b_t, z_t\right) \mid z_0 = z, b_0 = b \right\} \tag{15}$$

Applying the Feynman-Kac formula, we obtain the following representation as a partial differential equation (PDE)

$$\begin{aligned}\rho V_{def}(b,z) &= \max_{\pi} u_{def}(z) - x(\pi, e^z - \epsilon\left(e^z\right)) - \pi b \frac{\partial V_{def}}{\partial b} - \mu z \frac{\partial V_{def}}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V_{def}}{\partial z^2} \\ &\quad + \chi\left(V\left(\theta b, z\right) - V_{def}(b,z)\right),\end{aligned} \tag{16}$$

where $u_{def}(z) = \frac{\left[e^z - \epsilon(e^z)\right]^{1-\gamma} - 1}{1-\gamma}$ and the term $-\pi b$ is the reduction in the real value of debt due to inflation. The first order condition is $\pi\left(b,z\right) = -\frac{1}{\psi(e^z - \epsilon(e^z))} b \frac{\partial V_{def}}{\partial b}$.

2.4 Foreign investors and bond pricing

The government sells bonds to competitive risk-neutral foreign investors that can invest elsewhere at the risk-free real rate $\bar{r}$. As explained before, during repayment spells bonds pay a coupon rate δ and are amortized at rate λ. But following a default (at some time $\tilde{t}$), and during the exclusion period of the government, investors receive no payments. Once the exclusion/renegotiation period ends (at time $\tilde{t} + \tau$), investors recover a fraction θ of the nominal value of each bond, such that their outstanding bonds will carry a market price $Q\left(\theta b_{\tilde{t}+\tau}, z_{\tilde{t}+\tau}\right)$. Finally, investors discount future nominal payoffs with the accumulated inflation between the time of the bond purchase (say, $t = 0$) and the time such payoffs accrue: $\int_0^t \pi_s ds$, where $\pi_s = \pi\left(b_s, z_s\right)$. Taking all these elements together, the nominal price of the bond at time $t = 0$ is given by

$$Q(b,z) = \mathbb{E}\left[\int_0^T e^{-(\bar{r}+\lambda)t - \int_0^t \pi_s ds}\left(\lambda + \delta\right) dt + e^{-(\bar{r}+\lambda)T - \int_0^T \pi_s ds} Q_{def}(b_T, z_T) \mid b_0 = b, z_0 = z\right], \tag{17}$$

where b and z follow the laws of motion (4) and (1), respectively, given the optimal policies $c(b,z)$ and $\pi(b,z)$. The price of the bond upon default $Q_{def}(b,z)$ is

$$Q_{def}(b,z) = \mathbb{E}\left[\int_0^\infty \chi e^{-(\bar{r}+\chi)t - \int_0^t \pi_s ds} \theta Q\left(\theta b_t, z_t\right) dt\right].$$

Applying the Feynman-Kac formula, we obtain the following PDEs

$$\left(\bar{r} + \pi(b,z) + \lambda\right) Q(b,z) = (\lambda + \delta) + s\left(b,z\right)\frac{\partial Q}{\partial b} - \mu z \frac{\partial Q}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 Q}{\partial z^2}, \text{ if } d(b,z) = 0, \tag{18}$$

$$Q(b,z) = Q_{def}(b,z), \text{ if } d(b,z) = 1,$$

$$\left(\bar{r} + \pi(b,z)\right) Q_{def}(b,z) = -\pi b \frac{\partial Q_{def}}{\partial b} - \mu z \frac{\partial Q_{def}}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 Q_{def}}{\partial z^2} + \chi\left[\theta Q\left(\theta b, z\right) - Q_{def}(b,z)\right] \tag{19}$$

for all (b,z), where the drift function $s\left(b,z\right) \equiv s\left(b,z,c\left(b,z\right),\pi\left(b,z\right)\right)$ is given by (11).

Given a current nominal bond price $Q\left(b,z\right)$, the implicit nominal *bond yield* $r\left(b,z\right)$ is the discount rate for which the discounted future promised cash flows from the bond equal its price. The discounted future promised payments are $\int_0^\infty e^{-(r(b,z)+\lambda)t}\left(\lambda+\delta\right) dt = \frac{\lambda+\delta}{r(b,z)+\lambda}$. Therefore, the bond yield function is

$$r\left(b,z\right) = \frac{\lambda + \delta}{Q\left(b,z\right)} - \lambda. \tag{20}$$

The gap between the nominal yield $r(b,z)$ and the risk-free real rate $\bar{r}$, $r(b,z)-\bar{r}$, is called the sovereign *spread*. The spread is typically defined in the data as the difference between the return of a risky and a risk-free bond of similar maturities. In our case, and given the fact that under the assumption of risk-neutral investors the risk-free yield curve is flat, the long-term yield on risk-free debt is also $\bar{r}$. The spread can be decomposed into an *inflation premium*, $r(b,z)-r^*(b,z)$, and a *default premium*, $r^*(b,z)-\bar{r}$,where $r^*(b,z)$ is the yield of a domestic bond issued in foreign currency.[16]

2.5 Equilibrium

We define our equilibrium concept:

Definition 1 (MPE) *A* Markov Perfect Equilibrium *is a value function $V(b,z)$, a consumption policy $c(b,z)$, inflation policy $\pi(b,z)$, a default policy $d(b,z)$ and a bond price function $Q(b,z)$ such that:*

1. *Given prices Q, the value function V solves the government problem (10); the optimal inflation is π, the optimal consumption is c, and the optimal default policy is d.*

2. *Given the optimal inflation π, consumption c and default policy d, bond prices solve the pricing equation (18).*

The government takes the bond price function as given and chooses inflation and consumption (continuous policies) and whether to default or not (stopping-time) to maximize its value function. The investors take these policies as given and price government bonds accordingly. Equilibrium in the no inflation regime is defined analogously, with $\pi=0$ replacing the inflation policy function. The definition of Markov Perfect Equilibrium (MPE) is a particular case of a Markov equilibrium in continuous-time games. It is composed by a coupled system of two partial differential equations (PDEs): the HJB equation and the bond pricing equation.[17]

2.6 The inflationary bias

Even if a complete analytical characterization of equilibrium is out of our reach, it is worthwhile to provide an analytical insight before moving to the numerical analysis in the following sections.

Proposition 1 (Inflation bias) *Inflation is always positive at positive debt levels:*

$$\pi(b,z)>0, \textit{for all } b>0.$$

Proof. The policy function for consumption (equation 12) is $[c(b,z)]^{-\gamma}=\frac{-1}{Q(b,z)}\frac{\partial V}{\partial b}$. Given that $Q(b,z)>0$, consumption utility is well-defined and finite only if $\frac{\partial V}{\partial b}<0$. Using this in the inflation policy function (equation 13), we have $\pi(b,z)=-\frac{\partial V}{\partial b}\frac{b}{\psi(e^z)}>0$ for all $b>0$. ■

[16] The latter can be computed through equations (18)-(19) by setting $\pi=0$.

[17] See Başar and Oldser (1999) or Dockner at al. (2000) for references on continuous-time (also known as differential) game theory.

The result in Proposition 1 is reminiscent of the classical "inflationary bias" of discretionary monetary policy originally emphasized by Kydland and Prescott (1977) and Barro and Gordon (1983). In those papers, the source of the inflation bias is a persistent attempt by the monetary authority to raise output above its natural level. Here, by contrast, it arises from the existence of a positive stock of non-contingent nominal sovereign debt (such that $b > 0$) and from the welfare gains that can be achieved by reducing the real value of such nominal debt ($-\frac{\partial V}{\partial b} > 0$) at the expense of foreign investors. In this regard, it is more closely related to the one arising in analyses of discretionary monetary policy in models with nominal non-contingent debt, such as Díaz-Giménez et al. (2008), Martin (2009), and Niemann et al. (2013).

3 Quantitative analysis

3.1 Computation

Having laid out our theoretical model, we now use it in order to analyze its equilibrium properties. As we are not able to solve the model analytically, we thus resort to numerical solutions. To this end, we introduce a new numerical algorithm to analyze continuous-time default models, described in Appendix B.[18]

Our numerical algorithm is based on an augmented model which assumes that the government may only default when it receives an exogenous option to default. We assume that the option to default follows a Poisson process with arrival rate ϕ. This model nests the case of continuous default choice by taking the limit as the arrival rate tends to infinity, $\phi \to \infty$. The advantage of this formulation is that both the HJB equations in the repayment and autarky regions as well as the bond pricing equation can be efficiently solved using an upwind finite difference scheme similar to the one introduced in Achdou et al. (2021). The complete algorithm has a "two-loop" structure: the inner loop computes value functions and bond prices given a default policy and the outer loop updates the default policy.

Compared to discrete-time methods, working in continuous time has several advantages. First, the computational burden is reduced: while solving the discrete-time Bellman equation requires the computation of expectations over all possible future states, in the continuous-time Hamilton-Jacobi-Bellman equation expectations are replaced by the first- and second-order derivatives of the value function. Second, the ergodic distributions can be efficiently computed using the Kolmogorov Forward (KF) equation, thus making it unnecessary to use more time-consuming and less precise methods such as Monte Carlo simulation, as typically done in

[18] Analytical solutions are seldom found in Markov Perfect equilibrium models, not even in the deterministic case. In fact, in the deterministic case of Markov Perfect Equilibrium, even the existence of a solution is not guaranteed in most cases, as discussed in Bressan (2010, Section 5). The stochastic case typically has a solution, but very restrictive assumptions (e.g. linear-quadratic structures) need to be imposed in order to be able to find it analytically; see e.g. the examples in Dockner at al. (2000). In a more stylized model of optimal default, for instance, Bressan and Nguyen (2016) are able to prove the existence of a solution in the open-loop Nash equilibrium but not in the Markov Perfect one. In the latter, they can only show that, if a smooth solution exists, it should satisfy a nonlinear partial differential equation, a result analogous to the definition of equilibrium in our model.

discrete-time models.

3.2 Calibration

We calibrate our model to Brazil, for two reasons. First, it is a country with a significant share of its external debt issued in local currency (see, for instance, Ottonello and Perez, 2019), which is a prerequisite for debt inflation policies to be effective. Second, the Brazilian sovereign debt crisis of 2002-03 is a good example of situations in which high inflation and fears of outright default have gone hand in hand. [19] All parameters are expressed in annual terms, though the model is solved at monthly frequency. The parameters of the endowment process are set to $\mu = 0.045$ and $\sigma = 0.027$. They are estimated from the linearly detrended quarterly real GDP in Brazil for the period 1996-2019. The bond amortization parameter λ is set such that the Macaulay bond duration is 2.3 years, in line with the data provided by the Brazilian Treasury for its public debt (Tesouro Nacional, 2020).[20] Coupon payments δ are 6.1 percent, in line with the average coupon paid on Brazilian bonds in *reais*, measured by JP Morgan's GBI (Global Bond Index) of Brazilian bonds over the period 2004-2019 (2004 being the first available year). We assume log consumption utility, i.e. $\gamma = 1$. We set χ to replicate an average period of exclusion $1/\chi$ of 6.5 years , and the risk-free rate $\bar{r}$ to 0.04. These two values are the annual counterparts of the quarterly ones in Chatterjee and Eyigungor (2012). The fraction θ of the nominal value of outstanding bonds recovered after default is set to 50%, consistently with the average value of haircuts in data analyzed by Benjamin and Wright (2013).[21]

Output costs take the form

$$\epsilon(y) = \max\left\{0, d_0 + d_1 y^2\right\},$$

as in Chatterjee and Eyigungor (2012). The remaining five parameters, $\{\rho, d_0, d_1, \psi, \zeta\}$, are chosen to fit nine targets from Brazilian data. These targets are the sample means from 2001 (first year for which data on sovereign spreads are available) to 2019, as well as the trough-to-peak variation and peak level in the 2002-2003 crisis episode, of the following three variables: (i) CPI inflation; (ii) the sovereign spread, computed as the difference between the yield of Brazilian bonds in *reais* –measured by JP Morgan's GBI– and the yield of equivalent US Treasury bonds; and (iii) the default premium, computed as the difference between the yield of Brazilian bonds in USD –measured by JP Morgan's EMBI– and that of US bonds. Table 1 summarizes the calibration.[22]

[19] In the period 1990-2003, for instance, Brazil issued 68 percent of its sovereign external debt in local currency, compared to 5 percent of Argentina. Arellano, Bai and Mihalache (2020) also consider Brazil in the context of a New Keynesian model with sovereign default risk.

[20] In the case of our exponentially decaying perpetual bonds, the steady-state Macaulay duration is $1/(\lambda + r_{ss})$.

[21] They find a mean of 51 percent an a median of 49 percent.

[22] The computational parameters are as follows. We consider 66 points in the debt space, ranging from 0 to 0.52, and 49 points in the income space, from -0.12 to 0.12.

Table 1. Baseline calibration

Parameter	Value	Description	Source / target
μ	0.045	Drift parameter output	Persistence Brazilian GDP
σ	0.027	Diffusion parameter output	Volatility Brazilian GDP
λ	0.264	Bond amortization rate	Macaulay duration 2.3 years
δ	0.061	Bond coupon rate	Average coupon payment
γ	1	1/IES	Log-utility
χ	0.1538	Reentry rate	Chatterjee and Eyigungor (2012)
$\bar{r}$	0.04	Risk-free real interest rate	Chatterjee and Eyigungor (2012)
θ	0.5	Fraction of debt after default	Benjamin and Wright (2013)
			(1) Sample average,
ρ	0.129	Household discount factor	(2) trough-to-peak increase in 2002-03
d_0	-0.323	Default cost parameter	and (3) peak level in 2002-03 crisis of
d_1	0.361	Default cost parameter	(i) inflation,
ψ	1.87	Scale of inflation costs	(ii) sovereign spread and
ζ	27.8	Procyclicality inf. costs	(iii) default premium

3.3 Equilibrium objects

We begin our analysis by studying the equilibrium in each policy regime.

Inflationary regime. The solid blue lines in Figure 1 show the inflationary (baseline) regime's value and policy functions in the repayment segment of the debt space, conditional on output being at its steady-state level ($y = 1$).[23] As shown by panel (a), the value function declines almost linearly with the debt burden, except for debt levels close to the default point, where the slope declines in absolute value. The reason is twofold. First, in contrast to discrete-time models, where a kink emerges at the default boundary, in continuous-time models the slope of the repayment value function at the default threshold must equal that of the default value function, $\partial V/\partial b = \partial V_{def}/\partial b$, a property known as 'smooth pasting'.[24] Second, the value function is flatter in the default region, because following the autarky period the government reenters international capital markets at a substantially lower debt level, so marginal debt changes have smaller welfare effects.

Armed with the value and bond price functions, one can then use the optimality conditions (13) and (12) to analyze the optimal inflation and consumption policies (second and third rows in Figure 1). According to equation (13), optimal inflation is proportional to the product of the slope of the value function (in absolute value) and the debt level: $\pi(b, \cdot) = \psi(\cdot)^{-1} b [-V_b(b, \cdot)]$.

[23] Figure 7 in Appendix D displays these equilibrium objects for alternative output levels.

[24] See e.g. Dixit and Pindyck, 1994, or Øksendal,1995, for a discussion of this property in continuous-time models.

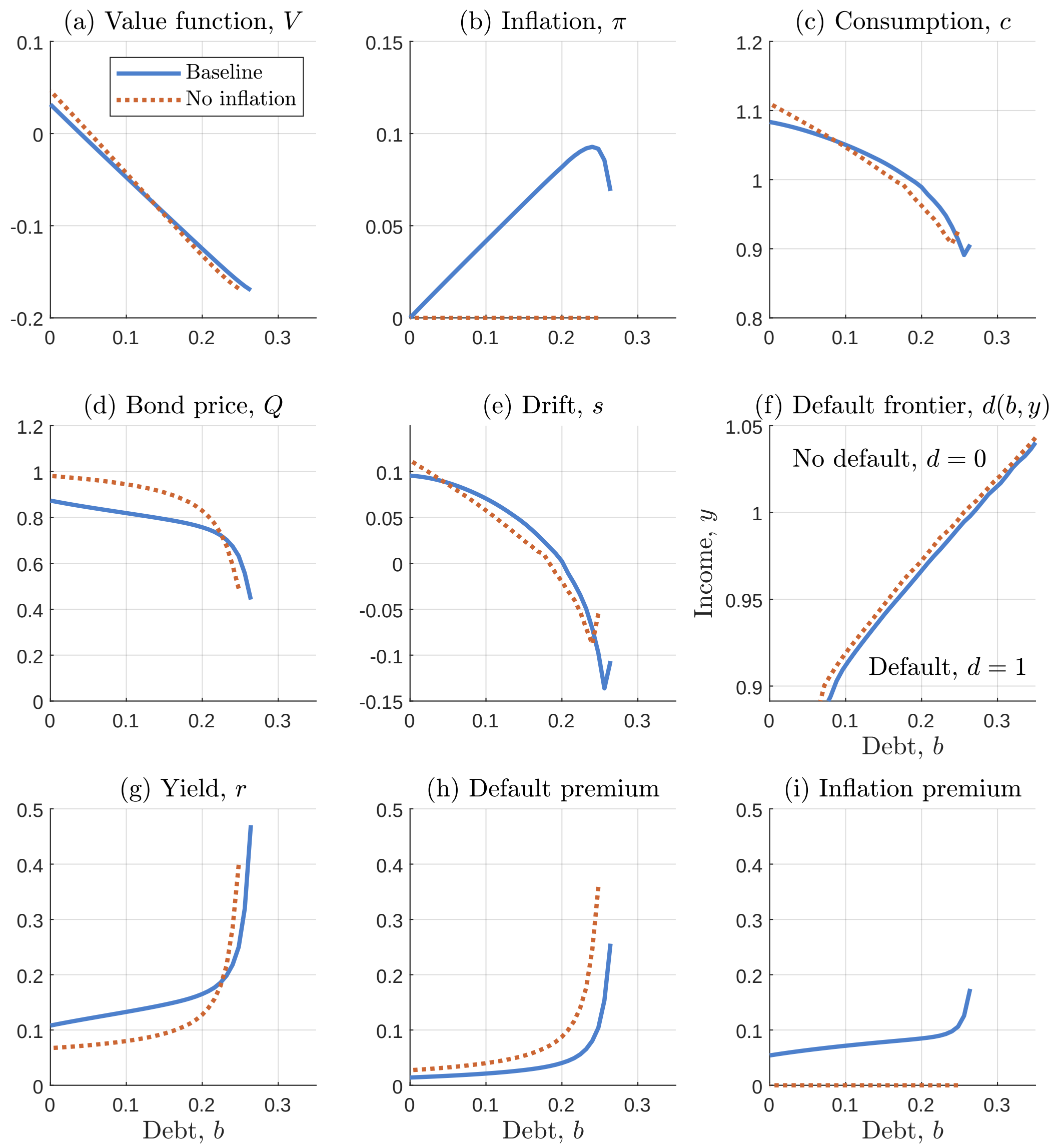

Figure 1: Equilibrium objects. The figure shows the equilibrium objects in the repayment segment of debt with $y = 1$. It also includes the default frontier (panel f). The 'baseline model' corresponds to the inflationary regime and the 'no inflation' to the no-inflation regime.

Since the value function is approximately linear for all debt levels except those very close to default, over that range the welfare gain per unit of debt reduction is roughly constant, and thus inflation increases approximately linearly with debt. This reflects the *inflationary bias* of monetary policy under discretion: as long as there is debt outstanding, the central bank will try to inflate it away.[25] In the vicinity of the default threshold, however, the flattening of the value function explained before implies that a marginal reduction in debt yields a lower welfare gain, and optimal inflation decreases slightly as a result. Intuitively, when default is really imminent, the incentives to inflate debt away are not so strong because the government anticipates the fact that, once default materializes, following the exclusion period it will enjoy a significant debt haircut anyway.

Consumption declines too with debt, reflecting –as per equation (12), which becomes $c(b,z) = \frac{Q(b,z)}{-\partial V/\partial b}$ under our log utility assumption– both the negative slope of the bond price function and the nearly constant slope of the value function in most of the debt space. As debt approaches the default threshold, the sharp decline in bond prices raises the cost of financing external deficits, which leads the government to reduce consumption more aggressively (panel c).

Panels (g) - (i) display the nominal yield, as well as the decomposition of the sovereign spread (i.e. the yield minus the riskless real rate $\bar{r}$) between inflation and default premia. The yield is proportional to the inverse of the bond price (see eq. 20) and hence it raises steadily for low and medium debt values, spiking as debt approaches the default frontier. This behaviour is mainly driven by the default premium, that is, by the compensation requested by investors for being exposed to the possibility of an outright default. The inflation premium is approximately flat for most of the state space, increasing in the vicinity of the default frontier as inflation rises.

Finally, panel (e) shows the drift function for debt accumulation. For moderate debt levels, the drift is mainly driven by the behavior of consumption, declining alongside the latter as debt increases. In the vicinity of default, however, the fall in the drift is reinforced by the steep decline in bond prices, which leads the government to drastically reduce –in fact, to turn negative– its pace of debt accumulation.

No-inflation regime. In order to better understand the consequences of discretionary inflation for macroeconomic and welfare outcomes, we compare our baseline model –which we refer to as the "inflationary regime"– with a counterfactual scenario in which inflation is zero in all states: $\pi(b,z) = 0$. This "no inflation regime", depicted by the red dashed lines in Figure 1, represents a scenario in which the government has effectively renounced the use of discretionary inflation, for instance by issuing real debt, or equivalently (in our model) foreign currency debt, in which case creating inflation is pointless in this framework; or by sticking to a credible inflation target.

Compared to the inflationary case, bond prices in the "no-inflation regime" are higher (i.e. yields are lower) for low and intermediate debt levels, reflecting the lack of any inflation

[25] Optimal inflation is exactly zero only when debt is zero, independently of output, as there is no incentive to create costly inflation when there is no debt to reduce.

premium associated with positive expected inflation during the life of the bond. However, as debt approaches the default threshold the situation reverses and bond prices are lower (yields are higher) than in the inflationary regime, due to the increase in the default premium.

Consumption is lower than in the inflationary regime for all debt levels except low ones. This reflects the fact that, at the margin, debt accumulation is more detrimental for welfare than in the inflationary scenario, where the government can use inflation as an additional debt-stabilization tool in response to aggregate shocks. As a result, the government can afford higher levels of consumption even if this means a faster pace of debt accumulation.

Default regions. The panels in Figure 1 reveal that, conditional on $y = 1$, the debt threshold for default is higher in the inflationary regime. In fact, this is true for any income level. Panel (f) in Figure 1 displays the default policy $d(b, y)$ in debt-income space. As is customary in (real) optimal sovereign default models, in the no-inflation regime the default frontier is upward-sloping in debt-output space: the government defaults when debt is sufficiently high (for given output) or output sufficiently low (for given debt). The same is true in the inflationary case. Importantly, the availability of the inflationary policy tool allows the government to *shift out* the default frontier. That is, inflation improves the sustainability of sovereign debt, in the sense that the government prefers repayment over outright default in states in which it would rather default if the inflationary tool was not available.

The intuition for the latter result as follows. Inflation represents a form of partial default on the real value of debt, one that is however more continuous and state-contingent than outright default. Indeed, outright default allows to reduce the debt burden in a discrete manner, but comes at the cost of an income loss and the inability to smooth consumption during the autarky period. In this sense, outright default is a comparatively blunt tool. Thus, when the economy lies close to the default frontier, the possibility of using inflation as a debt-stabilizing tool in response to aggregate income shocks allows the government to sustain higher and more stable levels of consumption, thus reducing the incentive to resort to outright default; this, in turn, reduces default premia and yields, which reinforces the positive effect on consumption. As a result, the government prefers repayment –in nominal terms– over outright default in a region of the debt-output space where it would choose the latter option if the inflationary policy was not available. That is, the default frontier shifts out *vis-à-vis* the no inflation regime.

At a fundamental level, the superior state-contingency of inflation as a debt stabilization tool is related to the natural asymmetry in the costs and benefits of outright default versus inflation as different forms of debt stabilization. Outright default triggers punishments on sovereigns that do not take place when inflation is used to erode the real value of nominal debt –at least as long as the *nominal* debt repayments are met. Likewise, unlike inflation, outright default allows for a discrete reduction in the real debt burden. These asymmetries are precisely what makes outright default a relatively blunt tool compared to inflation.

Our analysis above is based on comparing both regimes conditional on each state (b, y). A natural interpretation is to think of situations in which the government, faced with a debt burden b_t and an (exogenous) output level y_t, considers the possibility of a sudden, once-and-for-all regime change. For instance, a government that finances itself with nominal (domestic-currency) debt may consider converting it into real debt on a permanent basis, as a way of gaining anti-inflationary credibility. We may plausibly assume that, upon conversion, the government exchanges each nominal bond for a real bond with the same face value of one unit of domestic currency –with the difference that, from then on, the face value of the real bond grows at the same rate as inflation. Thus, the conversion turns B_t nominal bonds into the same number of real bonds with identical (initial) face value. The *real* face value of the new bonds at the time of the conversion would be $B_t/P_t = b_t$, i.e. exactly the same as in the just-abandoned nominal debt regime.[26] Therefore, when considering this regime change, the government would compare $V(b_t, y_t)$ with $V^{\pi=0}(b_t, y_t)$. The market price of debt would also change at the time of the regime change, from $Q(b_t, y_t)$ to $Q^{\pi=0}(b_t, y_t)$, as investors price the change in the future default and inflation policy. But the implications of this price jump are fully incorporated in all other equilibrium objects, including welfare.

The same considerations hold for the analysis of debt sustainability. Take e.g. the same regime change as before but in reverse: a government that finances itself with real (or foreign-currency) debt is experiencing sovereign debt stress –as represented by a state (b, y) close to the default frontier of the no inflation regime– and is considering converting its debt into nominal (domestic-currency) debt in order to be able to inflate it away at will. Again, the real face value of debt would not change at the time of the bond conversion, so neither would the economy's position in the state space. However, the regime change would shift out the default frontier, thus staving off (at least momentarily) the outright default. Hence, a change to the inflationary regime would improve debt sustainability.

Equilibrium uniqueness. Finally, an interesting issue is whether the equilibrium that we find is unique or if, instead, this model displays multiple equilibria à la Calvo (1988) in which expectations of future inflation translate into higher future nominal rates, which in turn result in higher inflation as debt increases. In order to search for the possible existence of multiple equilibria in the model, we try (radically) different initial guesses of both the inflation and the bond price functions, including guesses that entail high inflation levels –and low bond prices, i.e. high yields– across the state space. As shown in Figure 8 in Appendix D, our algorithm converges to the same equilibrium for very different initial guesses. Even though this cannot be taken as proof of uniqueness, it does give us some confidence that multiplicity is not a cause of concern in our analysis.

3.4 Comparative dynamics: impulse-responses

In order to further illustrate the comparative properties of both regimes, and particularly the role of inflation as a state-contingent shock absorber, we analyze the economy's dynamic response to income shocks. Figures 2 displays the generalized impulse-response functions fol-

[26]Notice that, in our (continuous-time) model, and under our assumption of quadratic inflation costs, the price level does not experience jumps. Therefore, at the time of the debt conversion, the price level would remain unchanged at P_t.

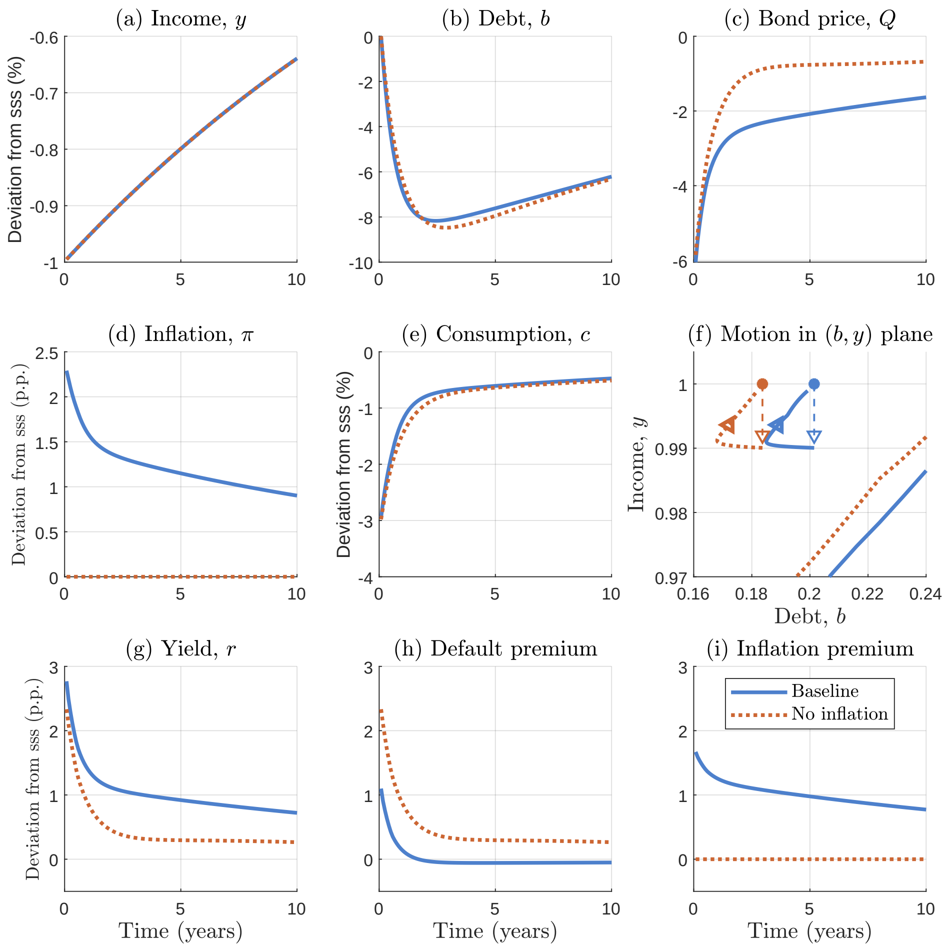

Figure 2: Generalized impulse response functions. The figure displays the generalized impulse response functions to a negative income shock when the economy starts at its stochastic steady state. The 'baseline model' corresponds to the inflationary regime and the 'no inflation' to the no-inflation regime.

lowing a negative income shock that decreases income by 1 percent (panel a), both for the baseline (blue solid lines) and the no-inflation regime (red dotted lines).[27] In each case, the initial condition is the stochastic steady state, $(b_0, y_0) = (b_{ss}, 1)$, where b_{ss} is defined as the debt level for which the drift is zero: $s(b_{ss}, 1) = 0$. This is the state to which all variables return asymptotically in the absence of further shocks.

Panel (b) displays the response of the debt stock. Let us start first with the *no-inflation* regime. On impact, the fall in income $y = e^z$ depresses bond prices (panel c), or equivalently, raises yields (panel g). This reflects the higher default risk as debt gets closer to the default frontier (panels h and f). Since financing external deficits through new debt issuance becomes more expensive, the government reduces the deficit, such that consumption (panel e) falls by more than income. The initial fall in aggregate spending dominates that in bond prices, and so debt is gradually reduced during the first three years following the shock. After that, the

[27] We use the standard Euler–Maruyama approximation of the law of motion of the state, given by $b_{t+\Delta t} - b_t = s(b_t, z_t)\Delta t$ and $z_{t+\Delta t} - z_t = -\mu z_t \Delta t + \sigma\sqrt{\Delta t}\varepsilon_{t+\Delta t}$, with $\varepsilon_{t+\Delta t} \sim N(0, 1)$. We choose a daily frequency, $\Delta t = 1/360$.

recovery in income allows the government to gradually return debt to its (stochastic) steady state.

Let us now turn to the *inflationary regime*. As shown in panel d, inflation increases on impact. Remember that inflation is given by $\pi(b, y) = \psi(y)^{-1} b [-V_b(b, y)]$. The fall in output does not substantially affect the slope in the value function on impact, but it does reduce the cost of inflation through the scale factor $\psi(y)$. This, coupled with the existence of a positive debt stock b, leads the government to raise inflation.

The increase in inflation allows the government to accelerate *ceteris paribus* the debt reduction process. Crucially, this allows the government to cushion the shock's impact on consumption, which indeed recovers faster than in the no-inflation case. By using inflation as a partial default tool, the government can also counteract to some extent the increase in default risk. Indeed, the default premium increases by less than in the no-inflation regime, although the actual yield increases by more as a result of the inflation premium.

In sum, partial default through inflation allows the government to flexibly respond to aggregate shocks and to absorb some of their impact on consumption. This allows the optimal policy to deliver a higher and more stable consumption stream, especially in the face of negative shocks that take the economy closer to the default frontier. Inflation of course entails a cost from the inflationary bias. In Section 3.6 we will analyze how this trade-off affects welfare outcomes.

3.5 Average behavior

So far we have compared both regimes in terms of their equilibrium behavior at each point of the state space. It is also interesting to compare their *average* behavior. In order to compute averages and other moments, we first need to solve for the ergodic debt-income distribution. For this purpose, it is useful to distinguish between (a) repayment spells and (b) the exclusion periods that follow each default. The stationary distributions conditional on being in a repayment spell and in a post-default autarky period are denoted by $g(b, y)$ and $g^{def}(b, y)$, respectively. They satisfy the *Kolmogorov Forward Equation* (KFE) described in Appendix B.

Panel (a) in Figure 3 represents $g(b, y)$ in the baseline inflationary regime as a heatmap. During repayment spells, the economy stays most of the time within a narrow region that runs roughly diagonally through the state space, from a high-income-high-debt zone to a low-income-low-debt one. Panel (b) displays, for both regimes, a slice of the respective joint density for $y = 1$. The inflationary regime is able to sustain slightly higher debt levels than the no-inflation one, consistently with the larger repayment region in that regime.

Having computed the ergodic debt-income distribution, we can then calculate a number of relevant moments in the model. Table 2 displays average values of key variables for both monetary regimes, as well as their empirical counterparts from Brazilian data for the period 2001-2019.[28] Notice first that, under the baseline inflationary regime, the model produces

[28] These moments are computed only over repayment spells. As explained in section 3.2, in our calibration we target a number of sample moments from the period 2001-2019. Since Brazil did not actually default over this

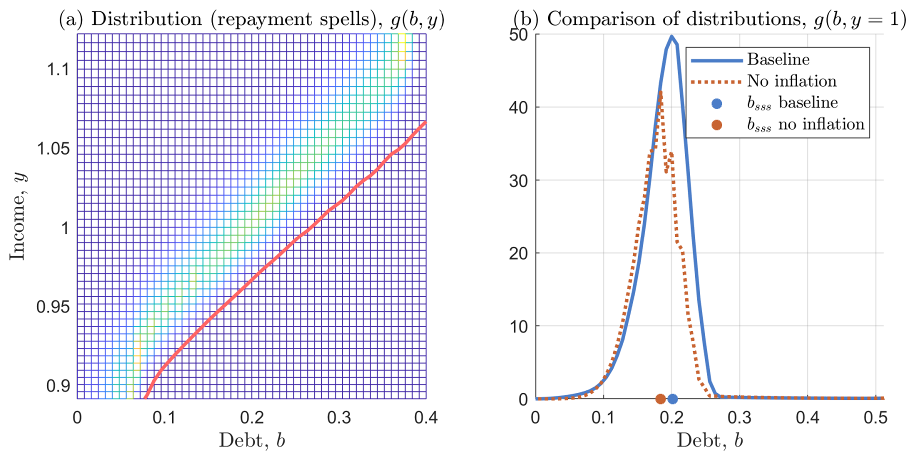

Figure 3: Stationary distribution. Panel (a) displays the stationary distribution $g(b, y)$ in the baseline regime. The red line is the default frontier. Panel (b) compares the stationary distributions in both regimes when $y = 1$. Circles indicate the stochastic steady states. The 'baseline model' corresponds to the inflationary regime and the 'no inflation' to the no-inflation regime.

a relatively good fit of the three sample moments used in the calibration: average inflation, average sovereign spreads, and the contribution of the default premium to sovereign spreads. This is despite the fact that five parameters were chosen to fit nine targets, including six targets related to the 2002-03 Brazilian debt crisis. By contrast, the model underpredicts the average ratio of external sovereign debt. This is due to the fact that this moment is not targeted and that models of this kind typically have a hard time producing empirically plausible sovereign debt ratios.

Table 2. Moments: data (2001-2019) vs model

			Policy regime	
Variable	units	Data	Baseline	No inflation
inflation, π	%	6.3	9.4	0
debt-to-GDP, b	%	9.3	20.1	21.5
spread, $r - \bar{r}$	%	11.2	12.5	6.2
inflation premium	%	6.7	9.2	0
default premium	%	4.6	3.3	6.2
spread (std. dev.)	%	4.6	6.8	3.7

Comparing the two regimes, several results emerge. First, the average sovereign spread is

period, when matching the model moments to their empirical counterparts the former are calculated conditional on the economy staying in the repayment region.

higher in the baseline regime, due to a sizable average inflation premium, and it is also more volatile. Second, the average default premium is lower than in the no-inflation regime, reflecting the key role of inflation for debt sustainability. Also, the average debt-to-GDP ratio is slightly higher without inflation. This reflects primarily the fact that, by continuing to inflate debt away during post-default exclusion spells, the government can reduce the real value of the debt burden by the time it reenters capital markets.

3.6 Welfare analysis

We now investigate the welfare consequences of discretionary inflation. To this aim, we start by decomposing the value function (eq. 9) as

$$V(b,z) = V_c(b,z) + V_\pi(b,z), \tag{21}$$

where

$$V_c(b,z) = \mathbb{E}\left\{\int_0^T e^{-\rho t}u(c_t)dt + e^{-\rho T}V_{def,c}(b_T, z_T) \mid b_0 = b,\ z_0 = z\right\}, \tag{22}$$

$$V_\pi(b,z) = \mathbb{E}\left\{-\int_0^T e^{-\rho t}x(\pi_t)\,dt + e^{-\rho T}V_{def,\pi}(b_T, z_T) \mid b_0 = b,\ z_0 = z\right\}, \tag{23}$$

and we have analogously decomposed the default value function as

$$V_{def,c}(b,z) = \mathbb{E}\left\{\int_0^\tau e^{-\rho t}u\left(e^{z_t} - \epsilon\left(e^{z_t}\right)\right)dt + e^{-\rho\tau}V_c(\theta b_\tau, z_\tau) \mid b_0 = b,\ z_0 = z\right\},$$

$$V_{def,\pi}(b,z) = \mathbb{E}\left\{-\int_0^\tau e^{-\rho t}x(\pi_t)\,dt + e^{-\rho\tau}V_\pi(\theta b_\tau, z_\tau) \mid b_0 = b,\ z_0 = z\right\}.$$

The components V_c and V_π represent the expected present-discounted stream of consumption utility flows and of inflation disutility flows, respectively.[29] Thus, V_π captures the welfare cost from the inflationary bias.

Figure 4 shows the welfare decomposition in both regimes, for $y = 1$. Notice first that the welfare costs from the inflationary bias are relatively stable across debt levels, consistently with the fact that households expect the government to inflate most of the time. The welfare costs of inflation can be as large as 0.4% of permanent consumption.[30]

The difference between the consumption component in both regimes, $V_c - V_c^{\pi=0}$, reflects the welfare benefits from the above-discussed role of inflation as a state-contingent partial default tool, namely allowing risk-averse households to enjoy a higher expected value of discounted consumption utility flows. As shown by the figure, this welfare benefit from inflation increases with debt, reaching a maximum at a level close to the default threshold of the inflation regime. The reason is that inflation is more effective at eroding the real value of debt the higher the debt

[29]See Appendix C for an explanation of how V_c and V_π can be computed. Obviously, $V_\pi = 0$ in the no-inflation regime.

[30]Under our assumption of log utility, the welfare costs of inflation in consumption equivalent terms is computed as $e^{\rho V_\pi(b,y)} - 1$.

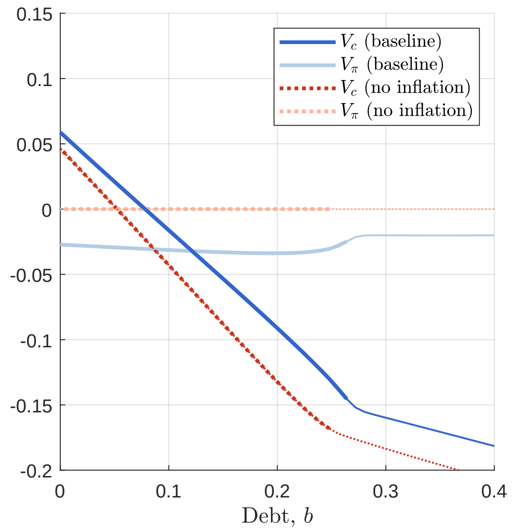

Figure 4: Welfare decomposition. The figure shows the value functions V_c and V_π in the repayment (thick line) and default (thin line) segments of debt with $y = 1$.

burden is. Indeed, inflation enters the debt drift function $s(b, \cdot)$ through the term πb, such that inflation is more effective at inflating debt away the higher debt is. As a result, it is when debt is close to the default point that inflation is most beneficial for the consumption utility component of household welfare. By contrast, the no-inflation regime achieves the highest welfare gains *vis-à-vis* the inflationary one at zero debt, because there the benefits from state-contingent inflation are at their minimum but the cost from the inflationary bias is already high.[31]

A more encompassing look at the welfare comparison of both regimes is provided by Figure 5, which portrays the region of the state space where the inflationary regime dominates the no-inflation one and how the welfare gain from inflation changes across states. The inflationary regime achieves better welfare outcomes for high debt levels. The welfare gains from inflation are largest in a region –marked in dark blue– of the state space close to the default frontier, in which such gains exceed 0.08% when expressed in consumption equivalent.[32] To understand why, we note that the pattern seen before in the welfare decomposition in Figure 4 for $y = 1$ carries over to a wide range of output levels around its steady state value: while the welfare cost of the inflationary bias V_π is fairly stable across debt levels, the consumption welfare gain of the inflationary regime, $V_c - V_c^{\pi=0}$, increases with debt and reaches a maximum in the vicinity of the default frontier, reflecting the fact that inflation is more effective at inflating debt away the higher the debt is. The maximum welfare gain is found for a point inside that dark-blue region and is as high as 0.12% of permanent consumption. By contrast, for zero debt the no-inflation regime achieves the highest welfare gains compared to the inflationary one, as represented by the dark brown area.

[31]Notice however that the point $b = 0$ is essentially never visited, as shown by the ergodic distribution in Figure 3.

[32]Under our assumption of log utility, such consumption equivalent is given by $e^{\rho\left(V(b,y) - V^{\pi=0}(b,y)\right)} - 1$.

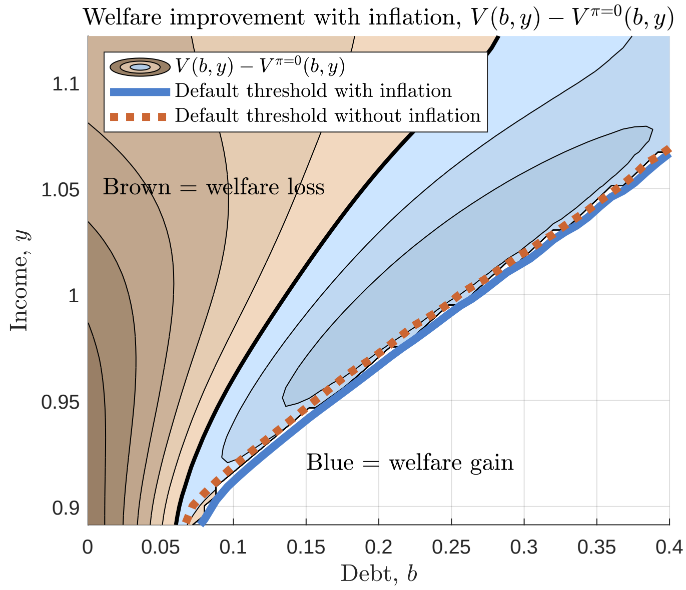

Figure 5: Isowelfare curves and default frontier. The blue region displays the isowelfare curves (b^{κ}, y^{κ}) such that $V(b^{\kappa}, y^{\kappa}) - V^{\pi=0}(b^{\kappa}, y^{\kappa}) = \kappa$. The blue region comprises the states in which $V(b,y) > V^{\pi=0}(b,y)$ and the red region $V(b,y) < V^{\pi=0}(b,y)$. The black line is the isowelfare with $\kappa = 0$. The solid blue line is the default frontier for the baseline regime and the dashed red line the default frontier for the no-inflation regime.

Average welfare. Beyond the conditional welfare analysis, it is also instructive to analyze the average welfare performance of both regimes. We use the ergodic distribution of the inflationary regime in order to average across conditional welfare gains. Under our log-utility assumption, we can express average welfare differences in consumption equivalent by solving for Γ in the following equation:

$$\int \int V(b,y)\, g(b,y)\, dbdy = \int \int \left[V^{\pi=0}(b,y) + \frac{\log(1+\Gamma)}{\rho} \right] g(b,y)\, dbdy.$$

We find that the inflationary regime generates an average welfare gain of 0.04% of permanent consumption. The interpretation is that, *under the veil of ignorance*, households living in an economy with nominal debt and a discretionary central bank would need to be compensated by 0.04% of their permanent consumption to switch to an economy with real (or foreign-currency denominated) debt.

This average gain from inflation is smaller than the conditional gains found before in a region close to the default frontier, and a third of the maximum conditional welfare gain (0.12%) found for a point inside that region. The reason is as follows. As shown in Figure 5, when the economy lies away from the default frontier, the no-inflation regime tends to outperform the inflationary one. This is because, at low debt levels, the cost from the inflationary bias exceeds the benefits from the state-contingency of inflation. Since the economy does spend some time away from the default frontier, when integrating across the state space the average welfare gains from inflation become smaller, as compared with those situations in which inflation is most useful

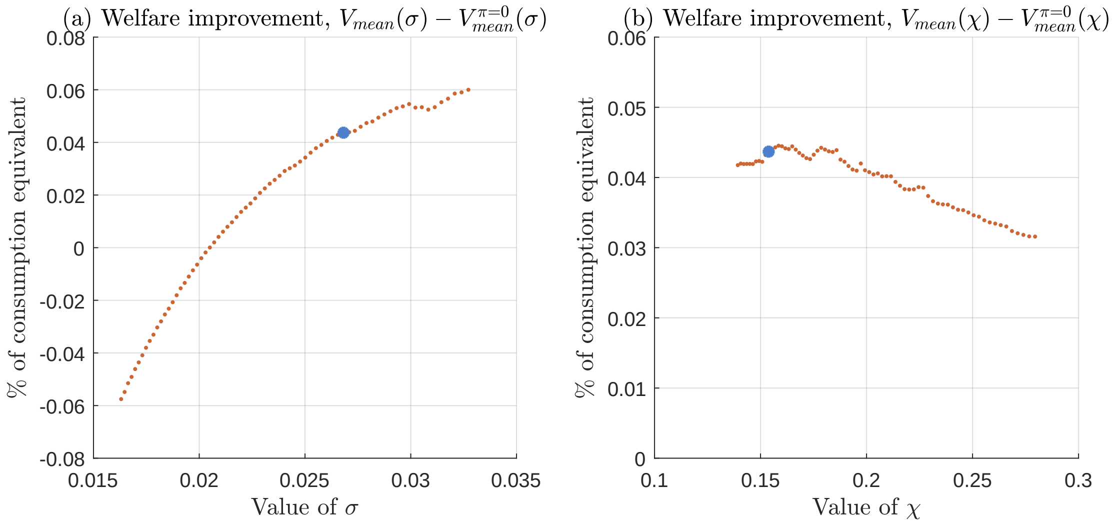

Figure 6: Average welfare difference between regimes as a function of parameters σ and χ. The welfare improvement $V_{mean}(\cdot) - V^{\pi=0}_{mean}(\cdot) = \int \left[V(b,y) - V^{\pi=0}(b,y)\right] g(b,y)\, dbdy$ is computed for different values of the parameters.

as a debt-stabilizing tool.[33]

Sensitivity analysis. We assess the sensitivity of our results to alternative values of the volatility of income shocks, σ, and of the average length of autarky spells, $1/\chi$. The volatility of income is the key parameter to assess the gains provided by inflation as a state-contingent partial default tool. Panel (a) of Figure 6 displays the average welfare gains from inflation for different values of the standard deviation of aggregate shocks. Such welfare gains increase monotonically with the size of aggregate shocks. This confirms the notion that inflation is valuable because it allows the government to respond to shocks in a more continuous, state-contingent manner. This state-contingency is more valuable in welfare terms the larger the amplitude of shocks hitting the economy.

The second advantage of inflation is that it enlarges the repayment region of the state space, thanks to its role as a partial default tool. The alternative is outright default, which allows for a considerable reduction in debt burden at a cost linked to the average duration of the ensuing autarky period ($1/\chi$). Therefore, the shorter the average length of the autarky period, the smaller the cost of outright default *vis-à-vis* that of inflation. Panel (b) of Figure 6 confirms this, by showing that the average welfare gain from inflation becomes smaller as χ increases.

[33]It is also interesting to compare welfare across different *constant* inflation rules. As shown in Figure 9 in Appendix D, the constant inflation rate that maximizes average welfare is actually not zero, but slightly positive (1.2%). The reason is that, even if constant inflation has no effect on real debt dynamics during repayment spells –as it is fully incorporated in bond prices–, it reduces the real value of debt during exclusion spells, as shown by equation (8). Figure 10 in the Appendix displays the value functions –net of the value funtion under zero inflation– for different constant inflation levels. As shown there, moderately positive constant inflation tends to outperform zero inflation across most states, except when output is sufficiently high and/or debt sufficiently low –i.e. precisely in situations in which outright default is more distant.

3.7 The Brazilian sovereign debt crisis of 2002-2003

Finally, we analyze the Brazilian sovereign debt crisis of 2002-2003 under the light of the model. We choose this episode because it provides a good example of a situation in which high inflation and fears of outright default go hand in hand, as they do in our model when the economy lies close to the default frontier. The narrative of the episode runs as follows. Brazil entered in recession in early 2001, according to the OECD Recession Indicators. This happened in the context of a global contraction after the dot-com bubble burst. During 2002, Brazilian sovereign spreads began to rise and remained high well into 2003, amid fears of a sovereign debt default similar to the one in Argentina in 2001. On the real side, Brazilian GDP experienced a sharp fall in late 2002 and early 2003 as well as a surge in inflation, which more than doubled in just a few quarters.

To simulate the crisis episode in our model, we set the initial value of debt-to-GDP equal to the one observed in 2002 and we feed the model with the sequence of income shocks such that aggregate income exactly replicates the evolution of detrended GDP in the data.[34] Table 3 compares the evolution of the key variables observed during the actual episode with their model-simulated counterparts. The model fits the episode-specific targets reasonably well,[35] considering the fact that, as explained in the section 3.1, five parameters are chosen to fit nine targets, including three moments over the whole sample period (2001-2019). The model captures three key dynamics in this episode. First, the fall in income produces an increase in sovereign spreads, driven both by default and inflation premia. Second, the model replicates well the increase in inflation. Third, in both cases the debt-to-GDP ratio is reduced.

In reality, three factors contributed to forestall the impending crisis: (i) a reduction in the debt-to-GDP ratio in 2003, thanks both to a large primary surplus and the rise in inflation; (ii) a stand-by credit by the IMF announced in August 2002, in exchange for a commitment to a public sector primary surplus of 3.75 percent of GDP in 2003 and a new strengthened Central Bank legislation aimed at curbing inflation (IMF, 2002); and (iii) the commitment by Luiz Inácio Lula da Silva ahead of the 2002 presidential election to adopt market-friendly policies and avoid an outright default if he won the election. Although our model can only capture the first factor, its predictions are approximately in line with the data, explaining how Brazil escaped outright sovereign default due to the reduction in the debt burden.

[34] Our data is quarterly for real GDP and annual for sovereign debt. In particular, our debt series is external public and publicly guaranteed debt (PPG), as provided by the World Bank's International Debt Statistics. Data for inflation, sovereign spread, and inflation and default premia are all monthly.

[35] As explained in the calibration section, the six episode-specific targets are: the trough-to-peak increase of inflation, the sovereign spread, and the default premium (displayed in Table 3) and the peak level of the three variables during the episode. The latter three statistics equal (20.7, 47.7, 19.5)% in the model, vs (17.2, 29.6, 19.0)% in the data.

Table 3. Through-to-peak variation during the 2002-2003 crisis: data vs model

Variable	units	Data	Baseline
GDP	%	-2.6	-2.6
inflation,π	pp	9.8	6.5
debt-to-GDP, b	pp	-1.7	-6.0
spread, $r - \bar{r}$	pp	15.7	26.2
inflation premium	pp	7.5	14.4
default premium	pp	11.8	11.8

Note: The table displays the through-to-peak variations of inflation, spread and premia, and the peak-to-through of GDP and debt-to-GDP ratio.

We now ask: how useful was the sharp rise in inflation at stabilizing Brazil's sovereign debt during the 2002-03 crisis? To answer this question, we compare the baseline model's predictions with those of a counterfactual no-inflation scenario, conditional on the same initial debt-to-GDP ratio and sequence of income shocks. This counterfactual scenario can be interpreted as one in which all Brazilian debt had been denominated in foreign currency, which in our model renders debt inflation policies ineffective. An alternative interpretation is that the central bank would have committed to maintaining inflation at zero, or equivalently in the model, to keep a peg against the U.S. dollar.

In such a counterfactual scenario, according to our model the Brazilian government would have actually *defaulted* in early 2003, as the sharp adjustment in aggregate spending alone would not have prevented the succession of negative income shocks from driving the economy into the default region. In welfare terms, we find that at the onset of the crisis, i.e. while the government is still in good credit in both scenarios, household welfare would have been lower in the counterfactual no-inflation scenario by 0.10% of permanent consumption. Once outright default ends up materializing in the no-inflation scenario, the welfare difference between the baseline and no-inflation scenarios increases to 0.26%. These results are a direct manifestation of the key insights explained in previous sections: it is in the face of negative output shocks that take the economy close to (outright) default that using inflation to stabilize debt and absorb the shocks' impact on consumption becomes more valuable in welfare terms.

4 Conclusions

This paper has addressed the consequences of discretionary inflation policy for sovereign debt sustainability and welfare, in a standard quantitative model of optimal sovereign debt default extended with nominal long-term debt and costly inflation. Two key results stand out. First, inflation enlarges the set of debt and income levels in which the government prefers repayment to default, relative to a counterfactual scenario where inflation is zero at all times. This is because inflation allows the government to erode the real value of debt. In this sense, inflation is a form of partial default, one that is more continuous and state-contingent than outright default, a comparatively blunt tool. Thus, the availability of the inflationary tool reducing the government's incentives for outright default.

Second, inflation provides quantitatively relevant welfare gains in situations of sovereign debt stress, such as the one experienced by Brazil in 2002. This reflects the following trade-off. Partial default through inflation allows the government to respond to aggregate shocks in a state-contingent manner, allowing households to enjoy a higher and more stable consumption stream. But discretionary inflation also entails a costly inflationary bias. While the latter cost is fairly stable over time, the benefits from the state–contingency of inflation are larger in the vicinity of default, because inflation is more effective as a debt-stabilizing tool when debt is high. As a result, inflation is most valuable near default. However, over the long run the welfare gains from discretionary inflation are more modest: because of the inflationary bias, the government creates costly inflation also when it is less useful as a debt-stabilizing tool.

References

[1] Achdou, Y., J.-M. Lasry, P.-L. Lions and B. Moll (2021), "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach," *The Review of Economic Studies*.

[2] Aguiar, M., M. Amador, E. Farhi and G. Gopinath (2013). "Crisis And Commitment: Inflation Credibility And The Vulnerability To Sovereign Debt Crises", mimeo.

[3] Aguiar, M., M. Amador, E. Farhi and G. Gopinath (2015). "Coordination And Crisis In Monetary Unions", *Quarterly Journal of Economics,* 130(4), 1727-1779.

[4] Aguiar, M., S. Chatterjee, S., H. Cole, and Z. Stangebye, (2016). "Quantitative Models of Sovereign Debt Crises," *Handbook of Macroeconomics*, Elsevier.

[5] Aguiar, M. and G. Gopinath (2006), "Defaultable debt, interest rates and the current account," *Journal of International Economics*, 69(1), 64-83.

[6] Araujo, A., M. Leon and R. Santos (2013), "Welfare analysis of currency regimes with defaultable debts," *Journal of International Economics*, 89(1), 143-153.

[7] Arellano, C. (2008). "Default Risk and Income Fluctuations in Emerging Economies," *American Economic Review*, 98(3), 690-712.

[8] Arellano, C., Y. Bai and G. Mihalache (2020). "Monetary Policy and Sovereign Risk in Emerging Economies (NK-Default)", mimeo.

[9] Arellano, C., X. Mateos-Planas, and J.-V. Ríos-Rull (2019). "Partial Default" NBER Working Paper No. 26076.

[10] Bacchetta, P., E. Perazzi and E. van Wincoop (2018). Self-Fulfilling Debt Crises: Can Monetary Policy Really Help?, *Journal of International Economics* 110, 119-134.

[11] Barro, R. J. and D. B. Gordon (1983). "A Positive Theory of Monetary Policy in a Natural Rate Model," *Journal of Political Economy*, 91(4), 589-610.

[12] Başar, T. and G. J. Olsder (1999), *Dynamic Noncooperative Game Theory,* Ed. Society for Industrial and Applied Mathematics. 2 edition.

[13] Benjamin, D. and M. L. J Wright. (2013). "Recovery Before Redemption? A Theory of Delay in Sovereign Default." Mimeo.

[14] Bianchi, J. and J. Mondragon (2018). "Monetary Independence and Rollover Crises," Working paper, Federal Reserve Bank of Minneapolis.

[15] Bornstein, G. (2020). "A Continuous Time Model of Sovereign Debt," *Journal of Economic Dynamics and Control*, Elsevier, 118(C).

[16] Bressan, A. (2010), *Noncooperative differential games*, lecture notes.

[17] Bressan, A. and K. T. Nguyen (2016), "An equilibrium model of debt and bankruptcy," *ESAIM; Control, Optimization and Calculus of Variations*, 22, 953–982.

[18] Calvo, G. A. (1983), "Staggered prices in a utility-maximizing framework," *Journal of Monetary Economics*, 12(3), 383-398.

[19] Calvo, G. A. (1988), "Servicing the Public Debt: The Role of Expectations," *American Economic Review*, 1988, 78, 647-661.

[20] Camous, A. and R. Cooper (2019), ""Whatever It Takes" Is All You Need: Monetary Policy and Debt Fragility." *American Economic Journal: Macroeconomics*, 11 (4), 38-81.

[21] Chatterjee, S. and B. Eyigungor (2012), "Maturity, Indebtedness, and Default Risk," *American Economic Review*, 102(6), 2674-99.

[22] Cole, H. and T. Kehoe (2000), "Self-Fulfilling Debt Crises," *Review of Economic Studies*, 67, 91-116.

[23] Corsetti, G. and L. Dedola (2016), "The Mystery of the Printing Press: Self-Fulfilling Debt Crises and Monetary Sovereignty," *Journal of the European Economic Association*, 14 (6), 1329–1371.

[24] Da-Rocha, J. M., E. L. Giménez and F. X. Lores, (2013), "Self-fulfilling crises with default and devaluation," *Economic Theory*, 53(3), 499-535.

[25] Diaz-Gimenez, J., G. Giovannetti, R. Marimon and P. Teles (2008). "Nominal Debt as a Burden on Monetary Policy," *Review of Economic Dynamics*, 11(3), 493-514.

[26] Dixit, A. and R. S. Pindyck (1994), *Investment under Uncertainty*, Princeton University Press.

[27] Dixon H. (2020). “How will pandemic debts be paid for?” *Reuters*, https://www.reuters.com/article/us-health-coronavirus-debt-breakingviews-idUSKBN2220Y1.

[28] Dockner, E. J., S. Jorgensen, N. V. Long and G. Sorger (2000), *Differential Games in Economics and Management Science,* Ed. Cambridge University Press.

[29] Du, W. and J. Schreger (2017), ”Sovereign Risk, Currency Risk, and Corporate Balance Sheets”, mimeo

[30] Eaton, J. and M. Gersovitz (1981), ”Debt with Potential Repudiation: Theoretical and Empirical Analysis,” *Review of Economic Studies*, 48(2), 289-309.

[31] Hatchondo, J. C. and L. Martinez (2009), ”Long-duration bonds and sovereign defaults,” *Journal of International Economics*, 79(1), 117-125.

[32] International Monetary Fund (2002). *Press Release: IMF Approves US$30.4 Billion Stand-By Credit for Brazil* https://www.imf.org/en/News/Articles/2015/09/14/01/49/pr0240

[33] Kelton, S. and E. Chancellor (2020). “Can governments afford the debts they are piling up to stabilise economies?,” *Financial Times*, https://www.ft.com/content/53cb3f6a-895d-11ea-a109-483c62d17528.

[34] Kydland, F. E. and E. C. Prescott. 1977. ”Rules Rather Than Discretion: The Inconsistency of Optimal Plans,” *Journal of Political Economy*, 85, 473-91.

[35] Martin, F. (2009). ”A positive theory of government debt”, *Review of Economic Dynamics*, 12(4), 608–631.

[36] Na, S., S. Schmitt-Grohe, M. Uribe, and V. Yue (2018): “The twin Ds: Optimal Default and Devaluation,” *American Economic Review*, 108(7), 1773–1819.

[37] Niemann, S. (2011). ”Dynamic monetary–fiscal interactions and the role of monetary conservatism,” *Journal of Monetary Economics*, 58(3), 234-247.

[38] Niemann, S., P. Pichler and G. Sorger (2013). ”Public debt, discretionary policy, and inflation persistence,” *Journal of Economic Dynamics and Control*, 37(6), 1097-1109.

[39] Øksendal, B. (1995), *Stochastic Differential Equations*. Springer.

[40] Ottonello, P. and D. Perez (2019). “The Currency Composition of Sovereign Debt,*” American Economic Journal: Macroeconomics*, 11(3), 174-208.

[41] Pham, H. (2009) *Continuous-time Stochastic Control and Optimization with Financial Applications*. Springer.

[42] Rebelo, S., N. Wang, and J. Yang (2021). "Rare Disasters, Financial Development, and Sovereign Debt." NBER Working Paper No. 25031.

[43] Reis, R. (2013). "The Mystique Surrounding the Central Bank's Balance Sheet, Applied to the European Crisis," *American Economic Review*, 103(3), 135-40.

[44] Rotemberg, J. J. (1982). "Sticky Prices in the United States," *Journal of Political Economy*, 90(6), 1187-1211.

[45] Röttger, J. (2019). "Discretionary Monetary and Fiscal Policy with Endogenous Sovereign Default", *Journal of Economic Dynamics and Control*, 105(C), 44-66.

[46] Sunder-Plassmann, L. (2020). "Inflation, default and sovereign debt: The role of denomination and ownership", *Journal of International Economics*, 127(C), .

[47] Tesouro Nacional (2020). "What is the Average Maturity of the Brazilian Public Debt?" Ministerio da Economia, Brazil. https://www.gov.br/fazenda/pt-br/assuntos/noticias/2005/r191005.

[48] Tourre, F. (2017). "Macro-Finance Approach to Sovereign Debt Spreads and Returns," mimeo.

[49] Woodford, M. (2001). "Fiscal Requirements for Price Stability," *Journal of Money, Credit and Banking*, 33(3), 669-728.

Online appendix (not for publication)

A. An economy with costly price adjustment

In this appendix, we lay out a model economy with the following characteristics: (i) firms are explicitly modelled, (ii) a subset of them are price-setters but incur a convex cost for changing their nominal price, and (iii) the social welfare function and the equilibrium conditions are the same as in the model economy in the main text.

Final good producer

In the model laid out in the main text, we assumed that output of the single consumption good Y_t is exogenous. Consider now an alternative setup in which the single consumption good is produced by a representative, perfectly competitive final good producer with the following Dixit-Stiglitz technology,

$$Y_t = \left(\int_0^1 y_{it}^{(\varepsilon-1)/\varepsilon} di \right)^{\varepsilon/(\varepsilon-1)}, \tag{24}$$

where $\{y_{it}\}$ is a continuum of intermediate goods and $\varepsilon > 1$. Let P_{it} denote the nominal price of intermediate good $i \in [0,1]$. The firm chooses $\{y_{it}\}$ to maximize profits, $P_t Y_t - \int_0^1 P_{it} y_{it} di$, subject to (24). The first order conditions are

$$y_{it} = \left(\frac{P_{it}}{P_t} \right)^{-\varepsilon} Y_t, \tag{25}$$

for each $i \in [0,1]$. Assuming free entry, the zero profit condition and equations (25) imply $P_t = ({}_0^1 P_{it}^{1-\varepsilon} di)^{1/(1-\varepsilon)}$.

Intermediate goods producers

Each intermediate good i is produced by a monopolistically competitive intermediate-good producer, which we will refer to as 'firm i' henceforth for brevity. Firm i operates a linear production technology,

$$y_{it} = Z_t n_{it}, \tag{26}$$

where n_{it} is labor input and $Z_t = e^{z_t}$ is productivity, where z_t follows equation (1) in the main text. At each point in time, firms can change the price of their product but face quadratic price adjustment cost as in Rotemberg (1982). Letting $\dot{P}_{it} \equiv dP_{it}/dt$ denote the change in the firm's price, price adjustment costs in units of the final good are given by

$$\Psi_t \left(\frac{\dot{P}_{it}}{P_{it}} \right) \equiv \frac{\psi(Y_t)}{2} \left(\frac{\dot{P}_{it}}{P_{it}} \right)^2 \tilde{C}_t, \tag{27}$$

where $\tilde{C}_t$ is aggregate consumption. Let $\pi_{it} \equiv \dot{P}_{it}/P_{it}$ denote the rate of increase in the firm's price. The instantaneous profit function in units of the final good is given by

$$\begin{aligned}
\Pi_{it} &= \frac{P_{it}}{P_t} y_{it} - w_t n_{it} - \Psi_t(\pi_{it}) \\
&= \left(\frac{P_{it}}{P_t} - \frac{w_t}{Z_t}\right)\left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} Y_t - \Psi_t(\pi_{it}), \qquad (28)
\end{aligned}$$

where w_t is the perfectly competitive real wage and in the second equality we have used (25) and (26). Without loss of generality, firms are assumed to be risk neutral and have the same discount factor as households, ρ. Then firm i's objective function is

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} \Pi_{it} di,$$

with Π_{it} given by (28). Notice that the firm's optimization *problem* is not affected by sovereign defaults, although of course default does affect the aggregate variables that enter the firm's problem (Y_t, P_t, etc.). The state variable specific to firm i, P_{it}, evolves according to $dP_{it} = \pi_{it} P_{it} dt$. We conjecture that the aggregate state relevant to the firm's decisions can be summarized by $(b_t, z_t, P_t) \equiv S_t$.[36] Then firm i's *value function* $J(P_{it}, S_t)$ must satisfy the following Hamilton-Jacobi-Bellman (HJB) equation,

$$\begin{aligned}
\rho J(P_i, S) &= \max_{\pi_i} \left\{ \left(\frac{P_i}{P} - \frac{w}{e^z}\right)\left(\frac{P_i}{P}\right)^{-\varepsilon} Y - \Psi(\pi_i) + \pi_i P_i \frac{\partial J}{\partial P_i}(P_i, S) \right\} \\
&\quad + \mu_S'(S) D_S J(P_i, S) + \frac{1}{2}\sigma_S'(S)\left(D_{SS} J(P_i, S)\right)\sigma_S(S),
\end{aligned}$$

where the vectors $(\mu_S(S), \sigma_S(S))$ collect the drift and diffusion terms, respectively, of the aggregate states S, and (D_S, D_{SS}) are the gradient and Hessian operators, respectively, with respect to S.[37] The first order and envelope conditions of this problem are (we omit the arguments of J to ease the notation),

$$\psi(Y)\pi_i \tilde{C} = P_i \frac{\partial J}{\partial P_i},$$

$$\begin{aligned}
\rho \frac{\partial J}{\partial P_i} &= \left[\varepsilon \frac{w}{Z} - (\varepsilon - 1)\frac{P_i}{P}\right]\left(\frac{P_i}{P}\right)^{-\varepsilon} \frac{Y}{P_i} + \pi_i \left(\frac{\partial J}{\partial P_i} + P_i \frac{\partial^2 J}{\partial P_i^2}\right) \\
&\quad + \frac{\partial}{\partial P_i}\left[\mu_S'(S) D_S J + \frac{1}{2}\sigma_S'(S)(D_{SS} J)\sigma_S(S)\right].
\end{aligned}$$

[36]In particular, we later show that in equilibrium $Y_t = Z_t$, whereas w_t and $\tilde{C}_t$ are also functions of (b_t, Z_t, P_t). The states P_t and b_t follow the same laws of motion as in the main text, equations (2) and (4) respectively, whereas $z_t = \log Z_t$ follows equation (1).

[37]In particular, $\mu_S(S) = [s(b, z), -\mu z, \pi P]'$, where $s(b, z)$ is the drift of a as defined in section 2 of the main text; and $\sigma_S(S) = [0, \sigma, 0]'$.

In what follows, we will consider a symmetric equilibrium in which all firms choose the same price: $P_i = P, \pi_i = \pi$ for all i. After some algebra, it can be shown that the above conditions imply the following pricing Euler equation,[38]

$$\begin{aligned}\left(\rho - \frac{\tilde{C}_b(b,z)\, s(b,z) - \tilde{C}_z(b,z)\,\mu z}{\tilde{C}(b,z)}\right)\pi(b,z) &= \frac{\varepsilon-1}{\psi}\left(\frac{\varepsilon}{\varepsilon-1}\frac{w}{e^z}-1\right)\frac{e^z}{\tilde{C}(b,z)} \\ &\quad + s(b,z)\,\pi_b(b,z) - \mu z \pi_z(b,z) + \sigma^2 F(S) \qquad (29)\end{aligned}$$

where $\tilde{C}(b,z)$ and $\pi(b,z)$ denote the equilibrium policy functions for total spending and inflation, and $F(S)$ is a function of the aggregate state capturing the effect of aggregate uncertainty (σ) on firms' pricing decision. Equation (29) determines the market clearing wage w as a function of S.

Households and the utility costs of inflation

The representative household's preferences are given by

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} \log\left(\tilde{C}_t\right) dt,$$

where $\tilde{C}_t$ is household consumption of the final good. Define *total real spending* as the sum of household consumption and price adjustment costs,

$$\begin{aligned} C_t &\equiv \tilde{C}_t + \int_0^1 \Psi_t(\pi_{it})\, di \\ &= \tilde{C}_t + \frac{\psi(Y_t)}{2}\pi_t^2 \tilde{C}_t, \qquad (30)\end{aligned}$$

where in the second equality we have used the definition of Ψ_t (eq. 27) and the symmetry across firms in equilibrium. Instantaneous utility can then be expressed as

$$\begin{aligned}\log(\tilde{C}_t) &= \log(C_t) - \log\left(1 + \frac{\psi(Y_t)}{2}\pi_t^2\right) \\ &= \log(C_t) - \frac{\psi(Y_t)}{2}\pi_t^2 + O\left(\left\|\frac{\psi(Y_t)}{2}\pi_t^2\right\|^2\right), \qquad (31)\end{aligned}$$

where $O(\|x\|^2)$ denotes terms of order second and higher in x. Expression (31) is the same as the utility function in the main text (eq. 6), up to a first order approximation of $\log(1+x)$ around $x = 0$, where $x \equiv \frac{\psi(Y_t)}{2}\pi^2$ represents the percentage of aggregate spending that is lost to price adjustment. For our baseline calibration, the latter object is relatively small even for relatively high inflation rates, and therefore so is the error in computing the utility losses from price adjustment. Therefore, the utility function in the main text (for $\gamma = 1$) provides a fairly accurate approximation of the welfare losses caused by inflation in the economy with costly

[38]The proof is available upon request.

price adjustment described here. We have solved the equilibrium implied by the exact inflation disutility function in the first line of equation (31) and found that the results are virtually identical to those in the main text.

As in the model in the main text, the government rebates to the household all the net proceedings from its international credit operations, denoted by $\tilde{T}_t$ in nominal terms. We assume that the household supplies one unit of labor input inelastically: $n_t = 1$. It also receives firms' profits in a lump-sum manner. Thus the household's nominal budget constraint is

$$P_t\tilde{C}_t = P_t w_t + P_t \int_0^1 \Pi_{it} di + \tilde{T}_t.$$

In the symmetric equilibrium, each firm's labor demand is $n_{it} = y_{it}/Z_t = Y_t/Z_t$. Since labor supply equals one, labor market clearing requires

$$\int_0^1 n_{it} di = Y_t/Z_t = 1 \Leftrightarrow Y_t = Z_t.$$

Therefore, in equilibrium output is simply equal to exogenous productivity Z_t. Each firm's real profits equal $\Pi_{it} = Y_t - w_t - \frac{\psi(Y_t)}{2}\pi_t^2\tilde{C}_t$. Using this in the household's budget constraint, we obtain

$$\tilde{T}_t = P_t\left(\tilde{C}_t + \frac{\psi(Y_t)}{2}\pi_t^2\tilde{C}_t - Y_t\right) = P_t\left(C_t - Y_t\right),$$

where in the second equality we have used (30).

Fiscal and monetary policy

The government maximizes household welfare subject to the laws of motion of the aggregate state variables. The default scenario is the same as in the main text, with one qualification: upon default and during the subsequent exclusion period, productivity equals $(Z_t - \epsilon(Z_t))$. This, together with the fact that in equilibrium $Y_t = Z_t$, implies that the default scenario is exactly as in the main text. It is then trivial to show that the government's maximization problem is exactly the same as in the main text, once we take into account that (i) the welfare criterion is the same (equation 9), and (ii) the law of motion of the debt ratio is the same (equation 4). As a result, the policy functions for inflation and primary deficit ratio will also be the same: $\pi_t = \pi(b_t, z_t)$, $c_t = c(b_t, z_t)$.

Notice finally that, since $Y_t = Z_t$, in equilibrium we have $C_t = Z_t \equiv C(b_t, z_t)$, and therefore $\tilde{C}_t = C(b_t, z_t)/[1 + \frac{\psi(e^{z_t})}{2}\pi(b_t, z_t)^2] \equiv \tilde{C}(b_t, z_t)$. Likewise, the pricing Euler equation derived above (equation 29) determines the market clearing wage given the aggregate state: $w_t = w(b_t, z_t, P_t)$. We thus verify our previous conjecture that (b_t, z_t, P_t) are the relevant aggregate states for firms.

Appendix B: description of the numerical algorithm

The augmented model with an option to default

Here we present an algorithm to compute the equilibrium of the model with inflation. Our numerical algorithm is based on an augmented model which assumes that the government may only default when it receives an exogenous option to default. We assume that the option to default follows a Poisson process with parameter ϕ, that is, there is a number of random times $\{\tilde{t}_i\}_{i=1}^{\infty}$ at which the government decides whether to continue repaying its debt or to default. If the arrival rate tends to infinite $\phi \to \infty$ the government can default at any point in time, whereas when $\phi = 0$ default is not available. Therefore for a large enough value of ϕ this specification nests the case of continuous default discussed in the main body of the text. The advantage of this method is that it prevents some numerical problems associated with the numerical solution of default problems in continuous time discussed below. Instead of the level of debt, we employ the level of net assets $a = -b$.

The HJB in this case results in

$$\begin{aligned} \rho V(a,z) &= \max_{c,\pi,d\in\{0,1\}} \frac{c^{1-\gamma}-1}{1-\gamma} - \frac{\psi}{2} y^{\zeta}\pi^2 + s(a,z,c,\pi)\frac{\partial V}{\partial a} - \mu z \frac{\partial V}{\partial z} \\ &\quad + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial z^2} + \phi d(a,z)\left[V_{def}(a,z) - V(a,z)\right]. \end{aligned} \tag{32}$$

Notice that the variational inequality has been replaced by a HJB equation including the term

$$\phi d(a,z)\left[V_{def}(a,z) - V(a,z)\right].$$

If the default policy d at the state (a,z) is one then the term indicates how in the case of the arrival of an option to default the value function jumps to the case with default, $V_{def}(a,z)$. The HJB for the default value (16) function remains the same.

Bond prices in this case are given by

$$(\bar{r} + \pi(a,z) + \lambda)\, Q(a,z) = (\lambda+\delta) + s(a,z)\frac{\partial Q}{\partial a} - \mu z\frac{\partial Q}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 Q}{\partial z^2} + \phi d(a,z)\left[Q_{def}(a,z) - Q(a,z)\right], \tag{33}$$

where the term $\phi d(a,z)\left[Q_{def}(a,z) - Q(a,z)\right]$ indicates that in the case of default the price of the bond is $Q_{def}(a,z)$. The advantage of this formulation, with a finite ϕ is that bond prices in the no-default region are never zero. There is a lower bound $(Q_{def}(a,z) = 0)$

$$Q^{\min}(a,z) = \frac{(\lambda+\delta)}{\bar{r} + \pi(a,z) + \lambda + \phi} > 0,$$

which can be made arbitrarily small by increasing ϕ. Having a non-zero value of Q avoids the problem that the drift function $s(a,z)$ (defined in equation 11) becomes infinite at the default frontier as it is a function of the inverse of the bond price.

For values of ϕ larger than 4 $-$ corresponding to an average default option per quarter $-$ the equilibrium objects remain constant up to the considered precision. In any case, we calibrate $\phi = 8$ to avoid any possible error associated with the numerical scheme.

Solution to the no default Hamilton-Jacobi-Bellman equation

The HJB equation is solved by a finite difference scheme following Achdou et al. (2020). It approximates the value function $V(a,z)$ on a finite grid with steps Δa and Δz : $a \in \{a_1, ..., a_I\}$, $z \in \{z_1, ..., z_J\}$. We use the notation $V_{i,j} := V(a_i, z_j)$, $i = 1, ..., I$; $j = 1, ..., J$. The derivative of V with respect to a can be approximated with either a forward or a backward approximation:

$$\frac{\partial V(a_i, z_j)}{\partial a} \approx \partial_{a,F} V_{i,j} := \frac{V_{i+1,j} - V_{i,j}}{\Delta a}, \tag{34}$$

$$\frac{\partial V(a_i, z_j)}{\partial a} \approx \partial_{a,B} V_{i,j} := \frac{V_{i,j} - V_{i-1,j}}{\Delta a}, \tag{35}$$

where the decision between one approximation or the other depends on the sign of the savings function $s_{i,j} = \left(\frac{\lambda+\delta}{Q(a_i,z_j)} - \lambda - \pi(a_i, z_j)\right) a_i + \frac{e^{z_j} - c_{i,j}}{Q(a_i,z_j)}$ through an "upwind scheme" described below. The derivative of V with respect to z is approximated using a forward approximation

$$\frac{\partial V(a_i, z_j)}{\partial z} \approx \partial_z V_{i,j} := \frac{V_{i,j+1} - V_{i,j}}{\Delta z}, \tag{36}$$

$$\frac{\partial^2 V(a_i, z_j)}{\partial z^2} \approx \partial_{zz} V_{i,j} := \frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{(\Delta z)^2}. \tag{37}$$

The HJB equation (32) is

$$\begin{aligned} \rho V(a,z) &= u(c) - x(\pi, e^z) + \left[\left(\frac{\lambda+\delta}{Q(a,z)} - \lambda - \pi\right) a + \frac{e^z - c}{Q(a,z)}\right] \frac{\partial V}{\partial a} - \mu z \frac{\partial V}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial z^2} \\ &\quad + \phi d(a,z)\left[V_{def}(a,z) - V(a,z)\right], \end{aligned}$$

where

$$\begin{aligned} c &= (u')^{-1}\left[\frac{1}{Q(a,z)}\frac{\partial V}{\partial a}\right] = \left[\frac{1}{Q(a,z)}\frac{\partial V}{\partial a}\right]^{-1/\gamma} \\ \pi &= -\frac{a}{\psi e^{\zeta z}}\frac{\partial V}{\partial a} = -\frac{a}{\psi e^{\zeta z}} c^{-\gamma} Q(a,z). \end{aligned}$$

The HJB equation is approximated by an upwind scheme

$$\begin{aligned} \frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta} + \rho V_{i,j}^{n+1} &= u(c_{i,j}^n) - x(\pi_{i,j}^n, e^{\zeta z_j}) + \phi d_{i,j}^n V_{def,i,j}^n + \partial_{a,F} V_{i,j}^{n+1} s_{i,j,F}^n \mathbf{1}_{s_{i,j,F}^n > 0} \\ &\quad + \partial_{a,B} V_{i,j}^{n+1} s_{i,j,B}^n \mathbf{1}_{s_{i,j,B}^n < 0} - \mu z_j \partial_z V_{i,j}^{n+1} + \frac{\sigma_z^2}{2}\partial_{zz} V_{i,j}^{n+1} - \phi d_{i,j}^n V_{i,j}^{n+1}, \end{aligned}$$

where

$$
\begin{aligned}
s^n_{i,j,F} &= \left[\left(\frac{\lambda+\delta}{Q^n_{i,j}}-\lambda-\frac{-a_i}{\psi}\partial_{a,F}V^n_{i,j}\right)a_i+\frac{e^{z_j}-(u')^{-1}\left[\frac{\partial_{a,F}V^n_{i,j}}{Q^n_{i,j}}\right])}{Q^n_{i,j}}\right], \\
s^n_{i,j,B} &= \left[\left(\frac{\lambda+\delta}{Q^n_{i,j}}-\lambda-\frac{-a_i}{\psi}\partial_{a,B}V^n_{i,j}\right)a_i+\frac{e^{z_j}-(u')^{-1}\left[\frac{\partial_{a,B}V^n_{i,j}}{Q^n_{i,j}}\right])}{Q^n_{i,j}}\right].
\end{aligned}
$$

This can be expressed as:

$$
\frac{V^{n+1}_{i,j}-V^n_{i,j}}{\Delta}+\rho V^{n+1}_{i,j}=u(c^n_{i,j})-x(\pi^n_{i,j},e^{\zeta z_j})+\phi d^n_{i,j}V^n_{def,i,j}+V^{n+1}_{i-1,j}\varrho_{i,j}+V^{n+1}_{i,j}\beta_{i,j}+V^{n+1}_{i+1,j}\eta_{i,j}+V^{n+1}_{i,j-1}\xi+V^{n+1}_{i,j+1}\varsigma_j, \tag{38}
$$

where

$$
\begin{aligned}
c^n_{i,j} &= (u')^{-1}\left[\frac{1}{Q^n_{i,j}}\left(\partial_{a,F}V^n_{i,j}\mathbf{1}_{s^n_{i,j,F}>0}+\partial_{a,B}V^n_{i,j}\mathbf{1}_{s^n_{i,j,B}<0}+Q^n_{i,j}u'\left(c^{0,n}_{i,j}\right)\mathbf{1}_{s^n_{i,j,F}<0,s^n_{i,j,B}>0}\right)\right], \quad (39)\\
\pi^n_{i,j} &= -\frac{a_i}{\psi e^{\zeta z_j}}\left(c^n_{i,j}\right)^{-\gamma}Q^n_{i,j}, \quad (40)\\
\varrho_{i,j} &= -\frac{s^n_{i,j,B}\mathbf{1}_{s^n_{i,j,B}<0}}{\Delta a}, \\
\beta_{i,j} &= -\frac{s^n_{i,j,F}\mathbf{1}_{s^n_{i,j,F}>0}}{\Delta a}+\frac{s^n_{i,j,B}\mathbf{1}_{s^n_{i,j,B}<0}}{\Delta a}+\frac{\mu z_j}{\Delta z}-\frac{\sigma^2}{(\Delta z)^2}-d^n_{i,j}\phi, \\
\eta_{i,j} &= \frac{s^n_{i,j,F}\mathbf{1}_{s^n_{i,j,F}>0}}{\Delta a}, \\
\xi &= \frac{\sigma^2}{2(\Delta z)^2}, \\
\varsigma_j &= \frac{\sigma^2}{2(\Delta z)^2}-\frac{\mu z_j}{\Delta z},
\end{aligned}
$$

where

$$
\begin{aligned}
c^{0,n}_{i,j} &= Q^n_{i,j}\left(\frac{\lambda+\delta}{Q(a_i,z_j)}-\lambda-\pi^{0,n}_{i,j}\right)a_i+e^{z_j}, \\
\pi^{0,n}_{i,j} &= \left(\frac{-a}{\psi,e^{\zeta z_j}}\partial_{a,F}V^n_{i,j}+\frac{-a}{\psi,e^{\zeta z_j}}\partial_{a,B}V^n_{i,j}\right)/2.
\end{aligned}
$$

The state constraint $a\le 0$ is enforced by setting $s^n_{I,j,F}=0$. Similarly, we impose $a\ge a^{\min}$ for a value of $a^{\min}$ large enough, which requires $s^n_{1,j,B}=0$. Therefore, the values $V^{n+1}_{0,j}$ and $V^{n+1}_{I+1,j}$ are never used. The boundary conditions with respect to z are

$$
\frac{\partial V(a,\underline{z})}{\partial z}=\frac{\partial V(a,\bar{z})}{\partial z}=0,
$$

as the process is reflected. At the boundaries in the j dimension, equation (38) becomes

$$
\begin{aligned}
\frac{V_{i,1}^{n+1}-V_{i,1}^{n}}{\Delta}+\rho V_{i,j}^{n+1} &= u(c_{i,1}^{n})-x(\pi_{i,j}^{n})+\phi d_{i,1}^{n}V_{def,i,1}^{n} \\
&\quad +V_{i-1,1}^{n+1}\varrho_{i,1}+V_{i,1}^{n+1}\left(\beta_{i,1}+\xi\right)+V_{i+1,1}^{n+1}\eta_{i,1}+V_{i,2}^{n+1}\varsigma_{1}, \\
\frac{V_{i,J}^{n+1}-V_{i,J}^{n}}{\Delta}+\rho V_{i,j}^{n+1} &= u(c_{i,J}^{n})-x(\pi_{i,j}^{n})+\phi d_{i,J}^{n}V_{def,i,J}^{n} \\
&\quad +V_{i-1,J}^{n+1}\varrho_{i,J}+V_{i,J}^{n+1}\left(\beta_{i,J}+\varsigma_{J}\right)+V_{i+1,J}^{n+1}\eta_{i,J}+V_{i,J-1}^{n+1}\xi_{J}.
\end{aligned}
$$

Equation (38) is a system of $I\times J$ linear equations which can be written in matrix notation as:

$$
\frac{\mathbf{V}^{n+1}-\mathbf{V}^{n}}{\Delta}+\rho\mathbf{V}^{n+1}=\mathbf{u}^{n}+\mathbf{A}^{n}\mathbf{V}^{n+1}, \tag{41}
$$

where the matrix $\mathbf{A}^n$ and the vectors $\mathbf{V}^{n+1}$ and $\mathbf{u}^n$ are defined by:

$$
\mathbf{A}^{n}=\begin{bmatrix}
\beta_{1,1}+\xi & \eta_{1,1} & 0 & \cdots & 0 & \varsigma_{1} & 0 & 0 & \cdots & 0 \\
\varrho_{2,1} & \beta_{2,1}+\xi & \eta_{2,1} & 0 & \cdots & 0 & \varsigma_{1} & 0 & \cdots & 0 \\
0 & \varrho_{3,1} & \beta_{3,1}+\xi & \eta_{3,1} & 0 & \cdots & 0 & \varsigma_{1} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \varrho_{I,1} & \beta_{I,1}+\xi & \eta_{I,1} & 0 & 0 & \cdots & 0 \\
\xi & 0 & \cdots & 0 & \varrho_{1,2} & \beta_{1,2} & \eta_{1,2} & 0 & \cdots & 0 \\
0 & \xi & \cdots & 0 & 0 & \varrho_{2,2} & \beta_{2,2} & \eta_{2,2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \varrho_{I-1,J} & \beta_{I-1,J}+\varsigma_{J} & \eta_{I-1,J} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \varrho_{I,J} & \beta_{I,I}+\varsigma_{J}
\end{bmatrix}, \tag{42}
$$

$$
\mathbf{V}^{n+1}=\begin{bmatrix}
V_{1,1}^{n+1} \\ V_{2,1}^{n+1} \\ \vdots \\ V_{1,2}^{n+1} \\ V_{2,2}^{n+1} \\ \vdots \\ V_{I-1,J}^{n+1} \\ V_{I,J}^{n+1}
\end{bmatrix}, \qquad
\mathbf{u}^{n}=\begin{bmatrix}
u(c_{1,1}^{n})-x(\pi_{1,1}^{n},e^{z_1})+\phi d_{1,1}^{n}V_{def,1,1}^{n} \\
u(c_{2,1}^{n})-x(\pi_{2,1}^{n},e^{z_1})+\phi d_{2,1}^{n}V_{def,2,1}^{n} \\
\vdots \\
u(c_{1,2}^{n})-x(\pi_{1,2}^{n},e^{z_2})+\phi d_{1,2}^{n}V_{def,1,2}^{n} \\
u(c_{2,2}^{n})-x(\pi_{2,2}^{n},e^{z_2})+\phi d_{2,2}^{n}V_{def,2,2}^{n} \\
\vdots \\
u(c_{I-1,J}^{n})-x(\pi_{I-1,J}^{n},e^{z_J})+\phi d_{I-1,J}^{n}V_{def,I-1,J}^{n} \\
u(c_{I,J}^{n})-x(\pi_{I,J}^{n},e^{z_J})+\phi d_{I,J}^{n}V_{def,I,J}^{n}
\end{bmatrix}.
$$

The system can in turn be written as

$$
\mathbf{B}^{n}\mathbf{V}^{n+1}=\mathbf{d}^{n},
$$

where $\mathbf{B}^n=\left(\frac{1}{\Delta}+\rho\right)\mathbf{I}-\mathbf{A}^n$ and $\mathbf{d}^n=\mathbf{u}^n+\frac{\mathbf{V}^n}{\Delta}$. $\mathbf{I}$ is the identity matrix. Matrix $\mathbf{B}^n$ is a sparse matrix, and the system can be efficiently solved in Matlab.

In order to avoid abrupt jumps in bond prices we combine it with a relaxation scheme such that, given a constant $\varkappa \in (0,1)$, if we denote the result of the linear system of equations (41) as $\hat{V}_{i,j}^{n+1}$, then

$$V_{i,j}^{n+1} = \varkappa \hat{V}_{i,j}^{n+1} + (1-\varkappa) V_{i,j}^{n}. \tag{43}$$

Solution to the default Hamilton-Jacobi-Bellman equation (no inflation)

The HJB equation in this case (16) with $\pi = 0$ is

$$\rho V_{def}(a,z) = u_{def}(z) - \mu z \frac{\partial V_{def}}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V_{def}}{\partial z^2} + \chi \left[V(\theta a, z) - V_{def}(a,z) \right],$$

which, using a finite difference method identical to the one described above (this time no upwind is necessary as there is no control), yields

$$\frac{V_{def,i,j}^{n+1} - V_{def,i,j}^{n}}{\Delta} + \rho V_{i,j}^{n+1} = u_{def}(z_j) + \chi V_{\theta i,j}^{n} + V_{def,i,j}^{n+1}\hat{\beta}_j + V_{def,i,j-1}^{n+1}\xi + V_{def,i,j+1}^{n+1}\varsigma_j,$$

where $\hat{\beta}_j = \frac{\mu z_j}{\Delta z} - \frac{\sigma^2}{(\Delta z)^2} - \chi$. $V_{\theta i,j}^{n}$ is the interpolated value corresponding to the notional grid point θi. This equation can be written in matrix notation as:

$$\frac{\mathbf{V}_{def}^{n+1} - \mathbf{V}_{def}^{n}}{\Delta} + \rho \mathbf{V}_{def}^{n+1} = \mathbf{u}_{def}^{n} + \mathbf{A}_{def}^{n}\mathbf{V}_{def}^{n+1}, \tag{44}$$

where

$$\mathbf{A}_{def}^{n} = \begin{bmatrix}
\beta_1+\xi & 0 & 0 & \cdots & 0 & \varsigma_1 & 0 & 0 & \cdots & 0 \\
0 & \beta_1+\xi & 0 & 0 & \cdots & 0 & \varsigma_1 & 0 & \cdots & 0 \\
0 & 0 & \beta_1+\xi & 0 & 0 & \cdots & 0 & \varsigma_1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \beta_1+\xi & 0 & 0 & 0 & \cdots & 0 \\
\xi & 0 & \cdots & 0 & 0 & \beta_2 & 0 & 0 & \cdots & 0 \\
0 & \xi & \cdots & 0 & 0 & 0 & \beta_2 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \beta_J+\varsigma_J & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \beta_J+\varsigma_J
\end{bmatrix},$$

$$\mathbf{V}_{def}^{n+1} = \begin{bmatrix} V_{def,1,1}^{n+1} \\ V_{def,2,1}^{n+1} \\ \vdots \\ V_{def,1,2}^{n+1} \\ V_{def,2,2}^{n+1} \\ \vdots \\ V_{def,I-1,J}^{n+1} \\ V_{def,I,J}^{n+1} \end{bmatrix}, \quad \mathbf{u}_{def}^{n} = \begin{bmatrix} u_{def}(z_1) + \chi V_{\theta 1,1}^{n} \\ u_{def}(z_1) + \chi V_{\theta 2,1}^{n} \\ \vdots \\ u_{def}(z_2) + \chi V_{\theta 1,2}^{n} \\ u_{def}(z_2) + \chi V_{\theta 2,2}^{n} \\ \vdots \\ u_{def}(z_J) + \chi V_{\theta(I-1),J}^{n} \\ u_{def}(z_J) + \chi V_{\theta I,J}^{n} \end{bmatrix}. \tag{45}$$

Solution to the default Hamilton-Jacobi-Bellman equation (optimal inflation)

The HJB equation in this case (16) with optimal inflation is

$$\rho V_{def}(a,z) = u_{def}(z) - x(\pi, e^z - \epsilon(e^z)) - \pi a \frac{\partial V_{def}}{\partial a} - \mu z \frac{\partial V_{def}}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V_{def}}{\partial z^2} + \chi\left[V(\theta a, z) - V_{def}(a,z)\right],$$

with a first order condition $\pi = -\frac{a}{\psi(e^z - \epsilon(e^z))}\frac{\partial V_{def}}{\partial a}$. Proceeding as in the case without default, we construct a matrix $\mathbf{A}^n_{def}$ equal to matrix $\mathbf{A}^n$ in (42) except for the fact that the drift now is

$$\begin{aligned} s^n_{def,i,j,F} &= \frac{a_i^2}{\psi\left(e^{\zeta z_j} - \epsilon\left(e^{\zeta z_j}\right)\right)}\partial_{a,F}V^n_{def,i,j}, \\ s^n_{def,i,j,B} &= \frac{a_i^2}{\psi\left(e^{\zeta z_j} - \epsilon\left(e^{\zeta z_j}\right)\right)}\partial_{a,B}V^n_{def,i,j}, \end{aligned}$$

and that element $\beta_{i,j} = -\frac{s^n_{i,j,F}\mathbf{1}_{s^n_{i,j,F}>0}}{\Delta a} + \frac{s^n_{i,j,B}\mathbf{1}_{s^n_{i,j,B}<0}}{\Delta a} + \frac{\mu z_j}{\Delta z} - \frac{\sigma^2}{(\Delta z)^2} - \chi$.

Vectors $\mathbf{V}^{n+1}_{def}$ and $\mathbf{u}^n_{def}$ are defined as in (15) except for the fact that utility is now $u_{def}(z_j) - x\left(\pi^n_{def,i,j}, e^{\zeta z_j}\right) + \chi V^n_{\theta i,j}$ with

$$\pi^n_{def,i,j} = -\frac{a_i}{\psi\left(e^{\zeta z_j} - \epsilon\left(e^{\zeta z_j}\right)\right)}\left(\partial_{a,F}V^n_{def,i,j}\mathbf{1}_{s^n_{i,j,F}>0} + \partial_{a,B}V^n_{i,j}\mathbf{1}_{s^n_{i,j,B}<0}\right).$$

Solution to the bond price equation

The bond price equation (18) is

$$Q(a,z)\left(\bar{r} + \lambda + \pi(a,z)\right) = (\lambda+\delta) + s(a,z)\frac{\partial Q}{\partial a} - \mu z\frac{\partial Q}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 Q}{\partial z^2} + \phi\mathbf{1}_{d(a,z)>0}\left[Q_{def}(a,z) - Q(a,z)\right]$$

which, using again a finite difference method identical to the ones already described above (this time no upwind is necessary either), yields

$$\begin{aligned} \frac{Q^{n+1}_{i,j} - Q^n_{i,j}}{\Delta} + \left(\bar{r} + \lambda + \pi^n_{i,j}\right)Q^{n+1}_{i,j} &= (\lambda+\delta) + \phi\mathbf{1}_{d_{i,j}>0}Q^n_{def,i,j} + Q^{n+1}_{i-1,j}\varrho_{i,j} \\ &\quad + Q^{n+1}_{i,j}\beta_{i,j} + Q^{n+1}_{i+1,j}\chi_{i,j} + Q^{n+1}_{i,j-1}\xi_j + Q^{n+1}_{i,j+1}\varsigma_j, \end{aligned}$$

where

$$\frac{\mathbf{Q}^{n+1} - \mathbf{Q}^n}{\Delta} + \left(\bar{r} + \lambda + \mathbf{\Pi}^n\right)\mathbf{Q}^{n+1} = \tilde{\mathbf{Q}}^{n+1}_{def} + \mathbf{A}^n\mathbf{Q}^{n+1}, \qquad (46)$$

where $\mathbf{\Pi}^n$ is a diagonal matrix with elements $\pi^n_{i,j}$ and

$$\mathbf{Q}^{n+1} = \begin{bmatrix} Q^{n+1}_{1,1} \\ Q^{n+1}_{2,1} \\ \vdots \\ Q^{n+1}_{1,2} \\ Q^{n+1}_{2,2} \\ \vdots \\ Q^{n+1}_{I-1,J} \\ Q^{n+1}_{I,J} \end{bmatrix}, \ \tilde{\mathbf{Q}}^{n+1}_{def} = \begin{bmatrix} (\lambda+\delta) + \phi \mathbf{1}_{d_{1,1}>0} Q^{n+1}_{def,1,1} \\ (\lambda+\delta) + \phi \mathbf{1}_{d_{2,1}>0} Q^{n+1}_{def,2,1} \\ \vdots \\ (\lambda+\delta) + \phi \mathbf{1}_{d_{1,2}>0} Q^{n+1}_{def,1,2} \\ (\lambda+\delta) + \phi \mathbf{1}_{d_{2,2}>0} Q^{n+1}_{def,2,2} \\ \vdots \\ (\lambda+\delta) + \phi \mathbf{1}_{d_{I-1,J}>0} Q^{n+1}_{def,I-1,J} \\ (\lambda+\delta) + \phi \mathbf{1}_{d_{I,J}>0} Q^{n+1}_{def,I,J} \end{bmatrix}.$$

We also combine it with a relaxation scheme such that, if we denote the result of the linear system of equations (46) as $\hat{Q}^{n+1}_{i,j}$, then

$$Q^{n+1}_{i,j} = \varkappa \hat{Q}^{n+1}_{i,j} + (1-\varkappa) Q^n_{i,j}. \tag{47}$$

Solution to the default bond price equation (no inflation)

The bond pricing equation in this case is

$$\bar{r} Q_{def}(a,z) = -\mu z \frac{\partial Q_{def}}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 Q_{def}}{\partial z^2} + \chi \left[\theta Q \left(\theta a, z \right) - Q_{def}(a,z) \right],$$

which, using a finite difference method identical to the one described above (this time no upwind is necessary as there is no control), yields

$$\frac{Q^{n+1}_{def,i,j} - Q^n_{def,i,j}}{\Delta} + \bar{r} Q^{n+1}_{i,j} = \chi \theta Q^n_{\theta i,j} + Q^{n+1}_{def,i,j} \hat{\beta}_j + Q^{n+1}_{def,i,j-1} \xi + Q^{n+1}_{def,i,j+1} \varsigma_j.$$

$Q^n_{\theta i,j}$ is the interpolated value corresponding to the notional grid point θi. This equation can be written in matrix notation as:

$$\frac{\mathbf{Q}^{n+1}_{def} - \mathbf{Q}^n_{def}}{\Delta} + \bar{r} \mathbf{Q}^{n+1}_{def} = \mathbf{q}^n_{def} + \mathbf{A}^n_{def} \mathbf{Q}^{n+1}_{def}, \tag{48}$$

where

$$\mathbf{Q}^{n+1}_{def} = \begin{bmatrix} Q^{n+1}_{def,1,1} \\ Q^{n+1}_{def,2,1} \\ \vdots \\ Q^{n+1}_{def,1,2} \\ V^{n+1}_{def,2,2} \\ \vdots \\ Q^{n+1}_{def,I-1,J} \\ Q^{n+1}_{def,I,J} \end{bmatrix}, \quad \mathbf{q}^n_{def} = \begin{bmatrix} \chi \theta Q^n_{\theta 1,1} \\ \chi \theta Q^n_{\theta 2,1} \\ \vdots \\ \chi \theta Q^n_{\theta 1,2} \\ \chi \theta Q^n_{\theta 2,2} \\ \vdots \\ \chi \theta Q^n_{\theta (I-1),J} \\ \chi \theta Q^n_{\theta I,J} \end{bmatrix}.$$

Solution to the default bond price equation (optimal inflation)

The bond pricing equation in this case is

$$(\bar{r} + \pi(a,z))\, Q_{def}(b,z) = -\pi a \frac{\partial Q_{def}}{\partial a} - \mu z \frac{\partial Q_{def}}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 Q_{def}}{\partial z^2} + \chi \left[\theta Q\left(\theta a, z\right) - Q_{def}(a,z)\right].$$

This equation can be written in matrix notation as:

$$\frac{\mathbf{Q}_{def}^{n+1} - \mathbf{Q}_{def}^{n}}{\Delta} + \left(\bar{r} + \mathbf{\Pi}^n\right) \mathbf{Q}_{def}^{n+1} = \mathbf{q}_{def}^{n} + \mathbf{A}_{def}^{n} \mathbf{Q}_{def}^{n+1}, \tag{49}$$

where matrix $\mathbf{A}_{def}^{n}$ is the one constructed in the computation of the value function with default above.

Complete algorithm

The algorithm to solve the model is based on two loops, an inner loop that finds the value functions and bond prices given the optimal default, and an outer one which computes the optimal default.

Outer loop. Begin with an initial guess $V_{i,j}^0 = u(ra_i + z_j)/\rho$, $V_{def,i,j}^0 = \left[u_{def}(1) + \chi V_{I,j}^0\right]/(\rho + \chi)$, $Q_{i,j}^0 = 1$, $d_{i,j} = 0$ and set $m = 0$. Then:

1. Given $d_{i,j}^m$,run the inner loop to find $V_{i,j}^{m+1}$, $V_{def,i,j}^{m+1}$, $Q_{i,j}^{m+1}$ using as an initial guess $V_{i,j}^m$, $V_{def,i,j}^m$, $Q_{i,j}^m$, $Q_{def,i,j}^m$.
2. Compute $d_{i,j}^{m+1} = 1_{V_{def,i,j}^{m+1} > V_{i,j}^{m+1}}$ according to (14).
3. If $d_{i,j}^{m+1}$ is close enough to $d_{i,j}^m$ stop. If not set $m := m+1$ and go to step 1.

Inner loop. Given a default policy $d_{i,j}^m = 0$, begin with an initial guess $V_{i,j}^0 = V_{i,j}^m$, $V_{def,i,j}^0 = V_{def,i,j}^m$, $Q_{i,j}^0 = Q_{i,j}^m$ and set $n = 0$. Then:

1. Compute $\partial_{a,F} V_{i,j}^n$, $\partial_{a,B} V_{i,j}^n$, $\partial_z V_{i,j}^n$ and $\partial_{zz} V_{i,j}^n$ using (34)-(37).
2. Compute $c_{i,j}^n$ and $\pi_{i,j}^n$ using (39) and (40), respectively.
3. Find $V_{i,j}^{n+1}$ solving the linear system of equations (41) plus the relaxation scheme (43).
4. Find $V_{def,i,j}^{n+1}$ solving the linear system of equations (44).
5. Find $Q_{i,j}^{n+1}$ solving the linear system of equations (46) plus the relaxation scheme (47).
6. If $V_{i,j}^{n+1}$ is close enough to $V_{i,j}^n$ and $Q_{i,j}^{n+1}$ is close enough to $Q_{i,j}^n$, stop. If not set $n := n+1$ and go to step 1.

Solution to the Kolmogorov Forward equation

Finally, we describe here the algorithm to solve the Kolmogorov Forward equation

$$0 = -\frac{\partial}{\partial a}\left[s\left(a,z\right)g(a,z)\right] + \frac{\partial}{\partial z}\left[\mu z g(a,z)\right] + \frac{\sigma^2}{2}\frac{\partial g}{\partial z^2} - \phi d(a,z)g(a,z) + \frac{\chi}{\theta}g^{def}\left(a/\theta,z\right), \quad (50)$$

$$0 = -\frac{\partial}{\partial a}\left[s^{def}\left(a,z\right)g^{def}(a,z)\right] + \frac{\partial}{\partial z}\left[\mu z g^{def}(a,z)\right] + \frac{\sigma^2}{2}\frac{\partial g^{def}}{\partial z^2} + \phi d(a,z)g(a,z) - \chi g^{def}\left(a,z\right), \quad (51)$$

where $s^{def}\left(a,z\right) \equiv -\pi a$. We solve the equation using an upwind finite difference scheme. We use the notation $g_{i,j} \equiv g(a_i, z_j)$. The solution in matrix form is

$$\begin{aligned} \mathbf{A}^{\mathbf{T}}\mathbf{g} + \frac{\chi}{\theta}\mathbf{g}_{\theta}^{def} &= 0, \\ \mathbf{A}_{def}^{\mathbf{T}}\mathbf{g}^{def} + \phi\mathbf{D}\mathbf{g} &= 0, \end{aligned}$$

where $\mathbf{g}_{\theta}^{def}$ is the matrix corresponding to the interpolated versions of $g_{\theta,i,j}^{def} = g^{def}(a_i/\theta, z_j)$ and

$$\mathbf{D}^n = \begin{bmatrix} d_{1,1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & d_{2,1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & d_{3,1} & 0 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & d_{I-1,J} & 0 \\ 0 & 0 & \cdots & 0 & 0 & d_{I,J} \end{bmatrix}.$$

We solve the system iteratively using a standard linear equation solver. Finally, we normalize the joint distribution $g + g^{def}$ to one.

Appendix C. Welfare decomposition

We first show how one can express the two components of the welfare decomposition (equations 22 and 23) recursively. We start by expressing the HJB equation of the repayment value function (equation 32) as

$$\begin{aligned} \left[\rho + \phi d(b,z)\right]V\left(b,z\right) &= u\left(c\left(b,z\right)\right) - \frac{\psi}{2}\pi\left(b,z\right)^2 + \mathbf{s}\left(b,z\right)\frac{\partial V}{\partial b} - \mu z\frac{\partial V}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial z^2} \\ &\quad + \phi d(b,z)V_{def}(z). \end{aligned}$$

Similarly, we express the HJB equation of the default value function (equation 16) as

$$\left(\rho + \chi\right)V_{def}(z) = u_{def}(z) - \mu z\frac{\partial V_{def}}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V_{def}}{\partial z^2} + \chi V\left(\theta a, z\right).$$

Using our postulated decomposition, $V\left(b,z\right) = V_c\left(b,z\right) + V_\pi\left(b,z\right)$ and $V_{def}\left(b,z\right) = V_{def,c}\left(b,z\right) + V_{def,\pi}\left(b,z\right)$, in the above two expressions, and ignoring function arguments except where needed, we obtain

$$\begin{aligned}
\left[\rho+\phi d\right]\left(V_c+V_\pi\right) &= u\left(c\right)-\frac{\psi}{2}\pi^2+\mathbf{s}\left(\frac{\partial V_c}{\partial b}+\frac{\partial V_\pi}{\partial b}\right)-\mu z\left(\frac{\partial V_c}{\partial z}+\frac{\partial V_\pi}{\partial z}\right)\\
&\quad+\frac{\sigma^2}{2}\left(\frac{\partial^2 V_c}{\partial z^2}+\frac{\partial^2 V_\pi}{\partial z^2}\right)+\phi d\left(V_{def,c}+V_{def,\pi}\right),
\end{aligned}$$

$$\begin{aligned}
\left(\rho+\chi\right)\left(V_{def,c}+V_{def,\pi}\right) &= u_{def}-\mu z\left(\frac{\partial V_{def,c}}{\partial z}+\frac{\partial V_{def,\pi}}{\partial z}\right)+\frac{\sigma^2}{2}\left(\frac{\partial^2 V_{def,c}}{\partial z^2}+\frac{\partial^2 V_{def,\pi}}{\partial z^2}\right)\\
&\quad+\chi\left[V_c\left(\theta a,z\right)+V_\pi\left(\theta a,z\right)\right].
\end{aligned}$$

We can then write

$$\left[\rho+\phi d\right]V_c=u\left(c\right)+\mathbf{s}\frac{\partial V_c}{\partial b}-\mu z\frac{\partial V_c}{\partial z}+\frac{\sigma^2}{2}\frac{\partial^2 V_c}{\partial z^2}+\phi d V_{def,c},$$

$$\left[\rho+\phi d\right]V_\pi=-\frac{\psi}{2}\pi^2+\mathbf{s}\frac{\partial V_\pi}{\partial b}-\mu z\frac{\partial V_\pi}{\partial z}+\frac{\sigma^2}{2}\frac{\partial^2 V_\pi}{\partial z^2}+\phi d V_{def,\pi},$$

$$\left(\rho+\chi\right)V_{def,c}=u_{def}-\mu z\frac{\partial V_{def,c}}{\partial z}+\frac{\sigma^2}{2}\frac{\partial^2 V_{def,c}}{\partial z^2}+\chi V_c\left(\theta a,z\right),$$

$$\left(\rho+\chi\right)V_{def,\pi}=0-\mu z\frac{\partial V_{def,\pi}}{\partial z}+\frac{\sigma^2}{2}\frac{\partial^2 V_{def,\pi}}{\partial z^2}+\chi V_\pi\left(\theta a,z\right).$$

These four value functions can be then be solved using finite-difference methods similar to those described in Appendix B.Appendix D. Additional figures

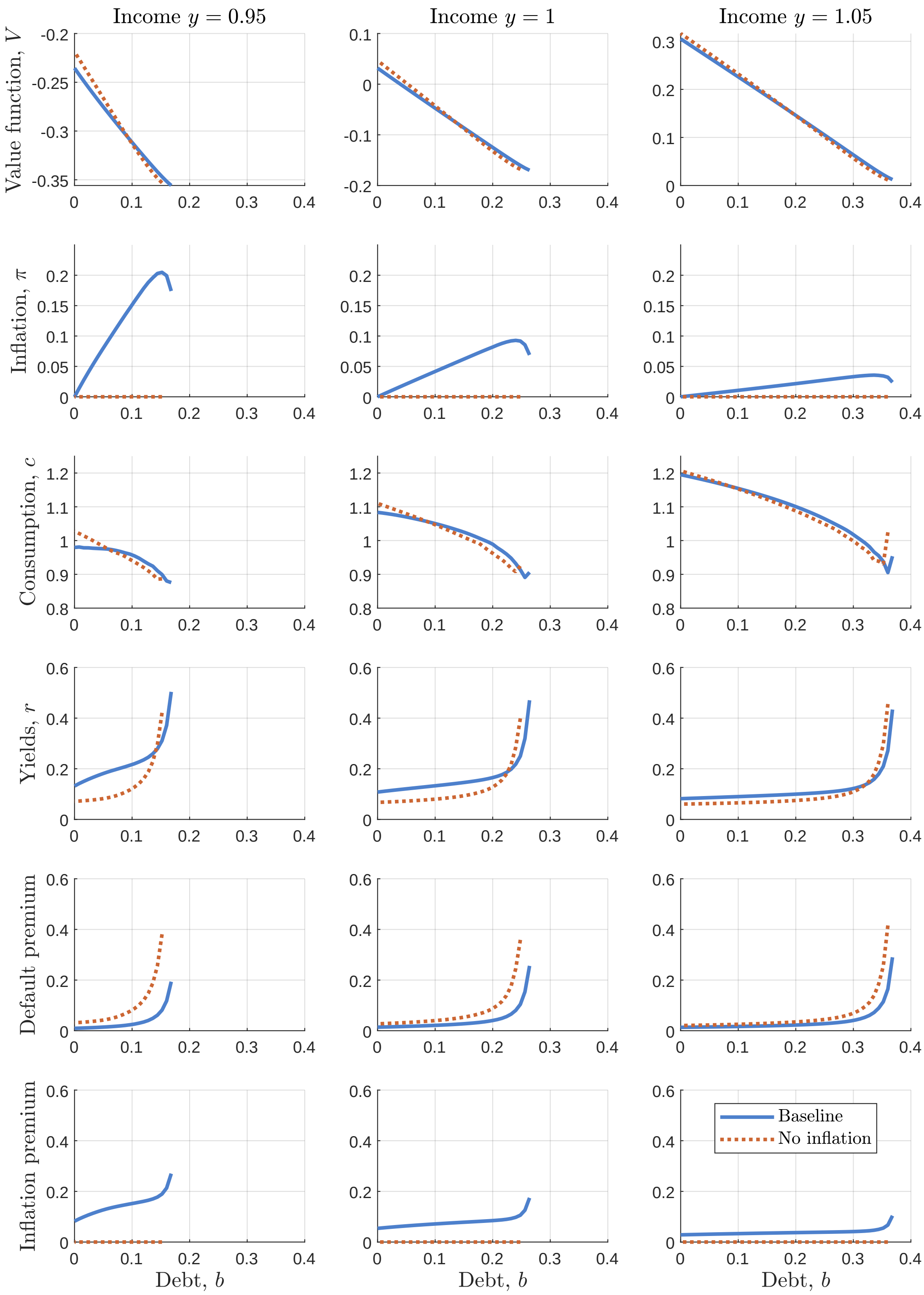

Figure 7: Equilibrium objects. The figure shows the equilibrium objects in the repayment (thick line) and default (thin line) segments of debt for different values y.

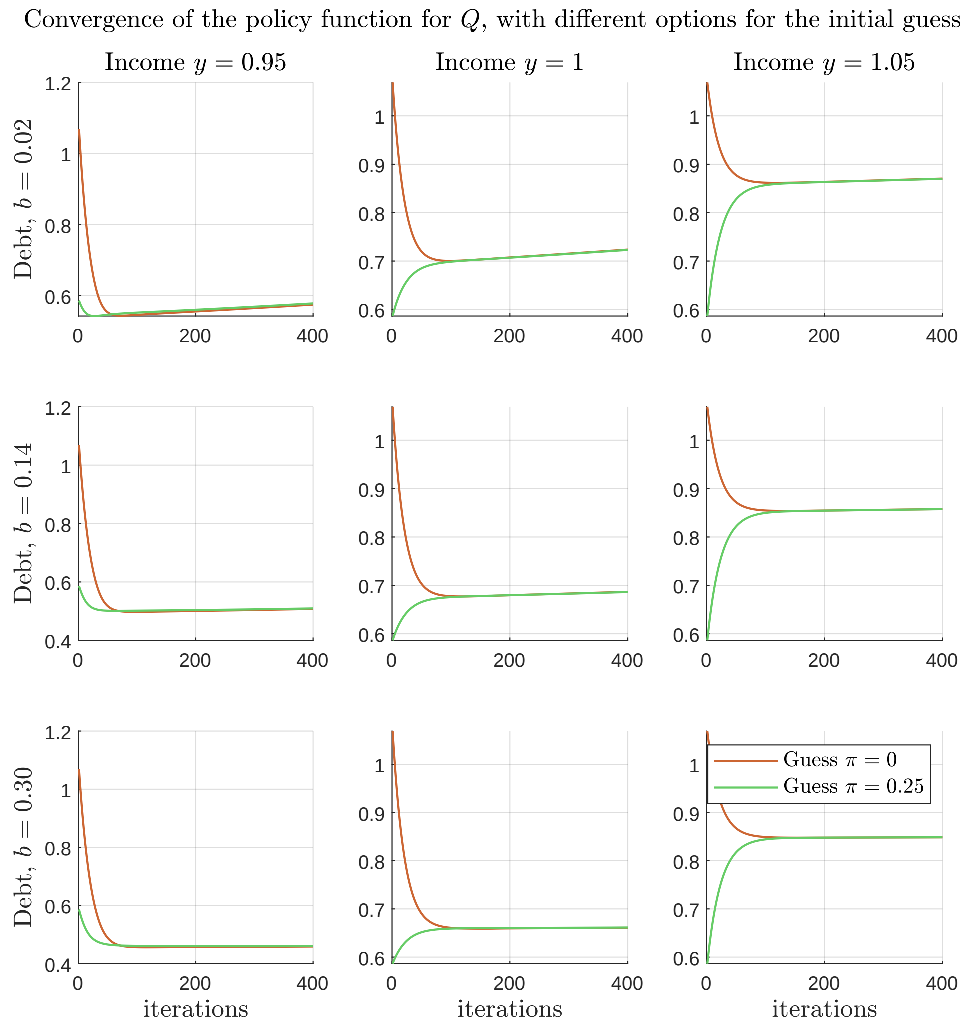

Figure 8: Convergence across different initial guesses. The figure shows the convergence of two initial guesses of the bond price schedule, consistent with inflation levels of $\pi = 0$ and $\pi = 0.25$.

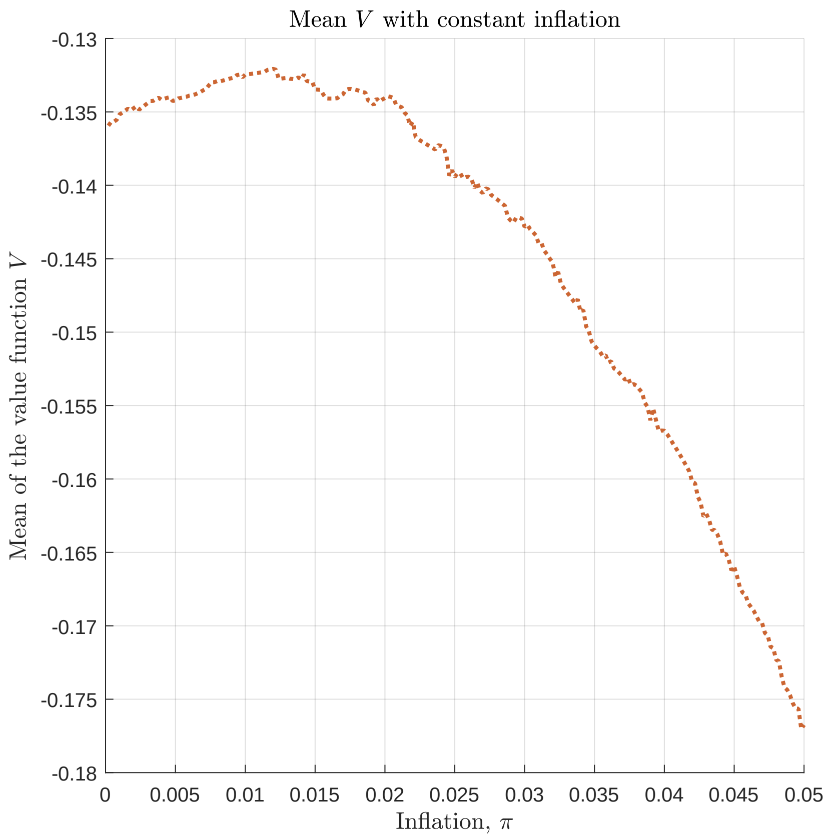

Figure 9: Average welfare as a function of the value of inflation in a constant inflation rule. The dotted red line is the average level of inflation, computed using the ergodic distribution, as a function of the value of inflation in a constant inflation rule.

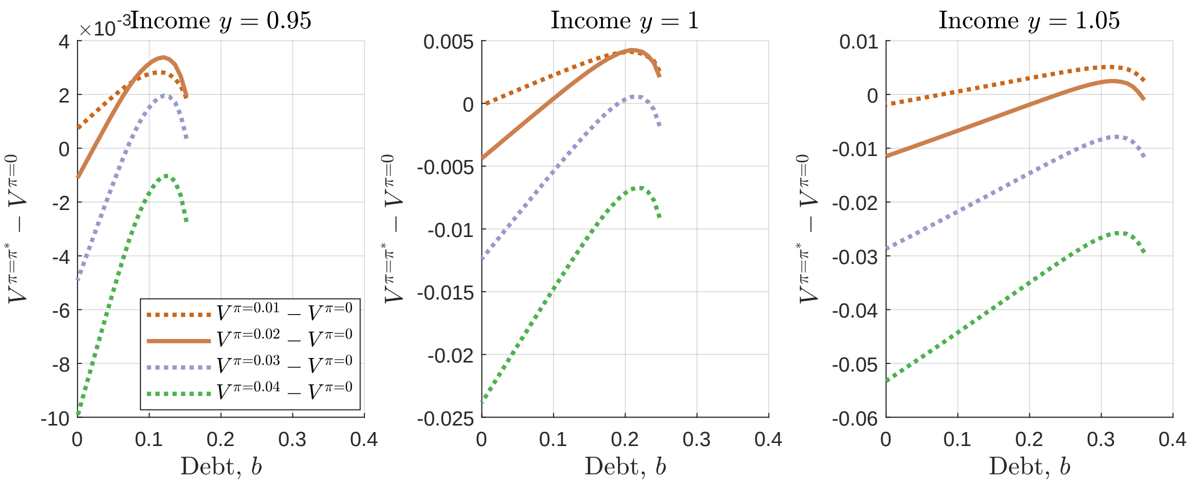

Figure 10: Welfare as a function of the value of inflation in a constant inflation rule. The figure shows the difference between the value functions $V_{\pi}^{\pi=\pi^*} - V_{\pi}^{\pi=0}$ for different values of constant inflation π^*.