Controlling a Distribution of Heterogeneous Agents

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Outline

Motivation

- Often questions in economics require computing the **optimal allocation** produced by a benevolent social planner
  - This is relatively straightforward with a representative agent...
  - ...but what about a continuum of **heterogeneous agents**?
- Other problems may also have an infinite-dimensional space state
  - Ex.: Oil extraction with a distribution of reserves, spatial AK models...

This paper

- Analyze **optimal control problems** with a continuum of **heterogeneous agents**
- Idiosyncratic risk (No aggregate shocks)
- Example: constrained-efficient equilibrium in the Aiyagari model with stochastic lifetimes

Framework of analysis

- **Continuous-time** setting
  - The dynamics of the distribution of agents are given by the **Kolmogorov Forward** (Fokker-Planck) equation
- Expressed as a deterministic control problem of a infinite-dimensional distribution
  - Subject to the aggregate – or market clearing (MC) – conditions

Preview of the results

1. Necessary conditions for a solution:
    1. Planner's **Hamilton-Jacobi-Bellman** (HJB) equation $\rightarrow$ social value $\neq$ private value
    2. Extended Mean Field Game (MFG) setting: HJB + KF + MC + LM (Lagrange multipliers)
2. Numerical algorithm
    1. Extends Achdou, Lasry, Lions and Moll (2015) to optimal control

Related literature

- Constrained-efficient problems in discrete-time models with incomplete markets and idiosyncratic risk
  - Dávila, Hong, Krusell and Ríos-Rull (2012)
- Optimal control problems in continuous time
  - Lucas and Moll (2014) or Afonso and Lagos (2015)
- Mean field control
  - Bensoussan, Frehse and Yam (2013)

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2. Competitive equilibrium
3. Planner's problem
4. Example
5. Computational algorithm
6. Ongoing and related research

Competitive equilibrium

Individual states

- Continuous-time infinite-horizon economy
  - Continuum of ex-ante identical agents $j \in [0,1]$
  - Stochastic lifetimes. Poisson death rate η
- Individual state $X_t^j \in \mathbb{R}^n$

$$dX_t^j = b\left(X_t^j, \mu(t, X_t^j), Z_t\right) dt + \sigma\left(X_t^j\right) dB_t^j$$

- Individual control $\mu(t, X_t^j) \in \mathcal{M}(x) \in \mathbb{R}^m$,

$$\mathcal{M}(x) \triangleq \{\mu : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m \ s.t. \ X(t) \in \Omega, \ \forall t \geq 0\},$$

  where $\Omega \subset \mathbb{R}^n$ (state constraint)
- Aggregate variable Z_t

Individual problem

Optimal value function $V(t, x)$

$$V(t, x) = \max_{\mu(\cdot)\in\mathcal{M}(x)} \mathbb{E}_t \left[\int_t^{\infty} e^{-(\rho+\eta)(s-t)} u(X(t), \mu) ds | X_t = x \right],$$

s.t. the law of motion of the state

HJB equation

$$\rho V\left(t,x\right)=\frac{\partial V}{\partial t}+\max_{\mu\in\mathcal{M}(x)}\left\{u(x,\mu)+\mathcal{A}V\right\},$$

where $\mathcal{A}$ is the **infinitesimal generator**

$$\mathcal{A}V=\sum_{i=1}^{n}b_i\left(x,\mu,Z\right)\frac{\partial V}{\partial x_i}+\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\left(\sigma(x)\sigma(x)^{\top}\right)_{i,k}}{2}\frac{\partial^2 V}{\partial x_i\partial x_k}-\eta V\left(t,x\right)$$

Aggregate distribution

The dynamics of the distribution $g(t,x)$ are given by the KF equation

$$\begin{aligned} \frac{\partial g}{\partial t} &= \mathcal{A}^* g + \eta \psi(x), \\ \int g(t,x)dx &= 1, \end{aligned}$$

where $\mathcal{A}^*$ is the **adjoint operator** of $\mathcal{A}$

$$\mathcal{A}^* g = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i g) + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial^2}{\partial x_i \partial x_k}\left[\left(\sigma\sigma^\top\right)_{i,k} g\right] - \eta g(t,x)$$

- Distribution of newborns: $\psi(x)$

Market-clearing conditions

- System of p equations:

$$Z_k(t) = \int f_k(x, \mu) g(t, x) dx, \quad k = 1, .., p.$$

Competitive equilibrium

Definition

A competitive equilibrium is a vector of aggregate variables $Z(t)$, a value function $V(t,x)$, a control $\mu(t,x)$ and a distribution $g(t,x)$ such that

1. Given $Z(t)$ and $g(t,x)$, $V(t,x)$ is the solution of the HJB equation and the optimal control is $\mu(t,x)$.
2. Given $\mu(t,x)$ and $Z(t)$, $g(t,x)$ is the solution of the KF equation.
3. Given $\mu(t,x)$ and $g(t,x)$, $Z(t)$ satisfy the market clearing conditions.

Outline

Planner's problem

- Assume instead a planner who chooses $\mu(t,x)$ to be applied to every agent
- The planner maximizes

$$J\left(g\left(0,\cdot\right)\right) \equiv \max_{Z,\ g,\mu \in \mathcal{M}(x)} \int_0^\infty \int e^{-\rho t} \omega(t,x) u(x,\mu) g(t,x) dx dt,$$

subject to law of motion of the distribution and to the market clearing conditions

A particular case

- If $\eta = 0$ and $\omega = 1$ (infinite lifetimes and utilitarian SWF)

$$J\left(g\left(0,\cdot\right)\right) = \int V(0,x)g(0,x)dx$$

Necessary conditions

- If a solution exists, then

$$J\left(g\left(0,\cdot\right)\right) = \int j(0,x)g(0,x)dx + \eta\int_0^\infty\int e^{-\rho t}j\left(t,x\right)\psi\left(x\right)dxdt,$$

where $j\left(t,x\right)$ is the **marginal social value function**

- The social value j satisfies

$$\rho j(t,x) = \frac{\partial j}{\partial t} + \max_{\mu\in\mathcal{M}(x)}\left\{\omega u + \sum_{k=1}^{p}\lambda_k(t)\left(f_k - Z_k\right) + \mathcal{A}j\right\},$$

Lagrange multipliers

The Lagrange multipliers $\lambda_k(t)$, $k = 1, .., p$:

$$\lambda_k(t) = -\int j \left\{ \sum_{i=1}^{n} \left[\frac{\partial^2 b_i}{\partial Z_k \partial x_i} g + \sum_{j=1}^{m} \frac{\partial^2 b_i}{\partial Z_k \partial \mu_j} \frac{\partial \mu_j}{\partial x_i} g + \frac{\partial b_i}{\partial Z_k} \frac{\partial g}{\partial x_i} \right] \right\} dx$$

- They reflect the '**shadow prices**' of the market clearing condition

Sketch of the proof

1. Given $\Phi \equiv [0,\infty) \times \mathbb{R}^2$, build the **Lagrangian** $\mathcal{L}(g, Z, \mu, j, \lambda)$ in the Hilbert space $L^2(\Phi)_{(\cdot,\cdot)_\Phi}$, with the scalar product

$$(f, g)_\Phi = \left\langle e^{-\rho t} f, g \right\rangle_\Phi = \int_\Phi e^{-\rho t} fg,$$

   where j, λ are the Lagrange multipliers of the KFE and MC conditions, respectively

2. Compute the **Gateaux derivative** with respect to each argument in the direction of an arbitrary $h(t,x) \in L^2(\Phi)_{(\cdot,\cdot)_\Phi}$

3. In the optimum, the Gateaux derivative should be **zero** for all $h \in L^2(\Phi)_{(\cdot,\cdot)_\Phi}$

- The Lagragian is

$$\mathcal{L}(g, Z, \mu, j, \lambda) = \left\langle e^{-\rho t}\omega u, g\right\rangle_\Phi + \left\langle e^{-\rho t} j, -\frac{\partial g}{\partial t} + \mathcal{A}^* g + \eta\psi\right\rangle_\Phi + \sum_{k=1}^{p}\left\langle e^{-\rho t}\lambda_k, (f_k - Z_k)\, g\right\rangle_\Phi,$$

- The Gateaux derivative of a functional $F(g)$ with respect to a function g is defined as

$$\delta F(g; h) = \lim_{\alpha\to 0}\frac{dF(g+\alpha h)}{d\alpha}$$

When is the competitive equilibrium socially optimal?

The competitive equilibrium equals the social optimum in the utilitarian sense $(\omega = 1)$ if

$$\sum_{k=1}^{p} \tilde{\lambda}_k(t)\left(f_k(x,\mu) - Z_k\right) = 0,$$

where $\tilde{\lambda}_k(t)$ are given by

$$\tilde{\lambda}_k(t) = -\int V \left\{ \sum_{i=1}^{n} \left[\frac{\partial^2 b_i}{\partial Z_k \partial x_i} g + \sum_{j=1}^{m} \frac{\partial^2 b_i}{\partial Z_k \partial \mu_j} \frac{\partial \mu_j}{\partial x_i} g + \frac{\partial b_i}{\partial Z_k} \frac{\partial g}{\partial x_i} \right] \right\} dx$$

Outline

An example

- Analyze the **optimal allocation** in a calibrated model of the US
  - Is there too much wealth inequality? Piketty (2014) or Atkinson (2015)
- In the competitive equilibrium, wealth should follow a **Pareto law**, as in the data

Model

- Continuous-time Aiyagari economy with **stochastic lifetimes** *à la* Blanchard-Yaari
- A benevolent social planner chooses the individual levels of consumption, while respecting all budget constraints
- With **infinite lifetimes** optimal allocation depends on the calibration (Dávila, Hong, Krusell and Ríos-Rull, 2012)
  - No ergodic distribution under the original Aiyagari's calibration
  - No Pareto distribution in the competitive equilibrium

Households

- Household's utility

$$\mathbb{E}_0\left[\int_0^\infty e^{-(\tilde{\rho}+\eta)t}\frac{c_t^{1-\chi}}{1-\chi}dt\right],$$

  with $\tilde{\rho} \triangleq \rho - (1-\chi)\,\gamma$ and γ is TFP growth
- Asset dynamics

$$da_t = \left(w_t z_t + \left(r_t - \gamma + \eta\right) a_t - c_t\right) dt,$$

- Borrowing limit

$$a_t \geq -\phi$$

- Idiosyncratic labor productivity

$$dz_t = \theta(\hat{z} - z_t)dt + \sigma dB_t,\ \ z_t \in [\underline{z}, \bar{z}]$$

Firms and market clearing condition

- Firms

$$\begin{aligned} r_t &= \alpha k_t^{\alpha-1} - \delta_K, \\ w_t &= (1-\alpha)\, k_t^{\alpha}. \end{aligned}$$

- Distribution dynamics

$$\frac{\partial g}{\partial t} = \mathcal{A}^* g\,(t, a, z) + \eta \delta_{a_0, z_0}$$

- Market clearing

$$k_t = \int_{-\phi}^{\infty} \int_{\underline{z}}^{\bar{z}} a g(t, a, z) da dz.$$

Pareto law in the competitive equilibrium

- The stationary wealth distribution follows a Pareto law

$$g(a) \sim a^{-(1+\zeta)},$$

with tail exponent

$$\zeta = \frac{\eta \chi}{(r - \gamma) - \tilde{\rho}}.$$

Constrained efficiency (Planner's problem)

$$J\left(g\left(0,\cdot\right)\right)=\max_{\left\{c\left(t,\cdot\right)\right\}_{t=0}^{\infty}}\int_{0}^{\infty}e^{-\rho t}\left[\int_{-\phi}^{\infty}\int_{\underline{z}}^{\bar{z}}u\left(c\right)g\left(t,a,z\right)dzda\right]dt,$$

subject to the law of motion of the aggregate distribution, to the factor prices and to the market clearing condition

Necessary conditions

The planner's HJB is

$$\tilde{\rho} j\left(t, a, z\right) = \frac{\partial j}{\partial t} + \max_{c} u\left(c\right) + \lambda(t)\left(a - k(t)\right) + \mathcal{A}j,$$

with a Lagrange multiplier

$$\lambda\left(t\right) = \frac{\alpha\left(1-\alpha\right)}{k\left(t\right)^{2-\alpha}} \int_{-\phi}^{\infty} \int_{\underline{z}}^{\bar{z}} j(t, a, z)\left(g(t, a, z) + a\frac{\partial g}{\partial a} - k\left(t\right) z \frac{\partial g}{\partial a}\right) dadz.$$

Outline

Computational algorithm

Steady-state

Given a constant $\theta \in (0,1)$, begin with an initial guess of k and λ:

1. Given k and λ, solve the HJB equation and obtain the social value function j and consumption c
2. Given c solve the KF equation and obtain the distribution g
3. Given c and g, compute $\tilde{k}$ using the MC conditions
    1. If $\tilde{k} \neq k$, return to 1
4. Given j, c and g, compute $\tilde{\lambda}$
    1. If $\tilde{\lambda} \neq \lambda$, set $\lambda := \theta\tilde{\lambda} + (1-\theta)\lambda$ and return to step 1.

Solution to the HJB equation

- Finite-difference "upwind" algorithm as in Achdou et al. (2015)
  - Converges to the **viscosity solution**
- Idea

$$\frac{\partial V(a_i, z_j)}{\partial a} \approx \partial_{a,F} V_{i,j} \equiv \frac{V_{i+1,j} - V_{i,j}}{\Delta a}, \text{ if } s^n_{i,j,F} > 0$$
$$\frac{\partial V(a_i, z_j)}{\partial a} \approx \partial_{a,B} V_{i,j} \equiv \frac{V_{i,j} - V_{i-1,j}}{\Delta a}, \text{ if } s^n_{i,j,B} < 0$$

Numerical solution of the HJB equation

- Applying finite differences, we obtain a finite-dimensional Bellman equation :

$$\begin{aligned} \tilde{\rho}\mathbf{V} &= \mathbf{u} + \mathbf{A}\mathbf{V}, \\ \mathbf{V} &= \mathbf{u} + (1-\tilde{\rho})\,\Pi\mathbf{V} \end{aligned}$$

where $\mathbf{A}$ is the discrete version of operator $\mathcal{A}$ and $\Pi = \mathbf{I} + \frac{1}{1-\tilde{\rho}}\mathbf{A}$

- This can be solved by **policy function iteration** with $\mathbf{c}^n = c\left(\mathbf{V}^n\right)$ and

$$\mathbf{V}^{n+1} = \left(\tilde{\rho}\mathbf{I} - \mathbf{A}^n\right)^{-1}\mathbf{u}^n$$

Solution to the KF equation

- As $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$ then the steady-state KF reduces to

$$\mathbf{A}^{\mathbf{T}}\mathbf{g}+\eta\delta_0 = \mathbf{0},$$

and the normalization

$$g_{i,j} = \frac{\hat{g}_{i,j}}{\sum_{i=1}^{I}\sum_{j=1}^{J} g_{i,j}\Delta a \Delta z}.$$

Advantages of continuous-time methods

1. The optimal policy is explicit (no need to compute max.)

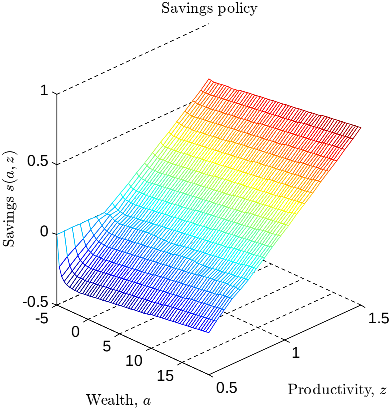
$$c\left(V\right) = \left(u'\right)^{-1}\left(\frac{\partial V}{\partial a}\right),$$

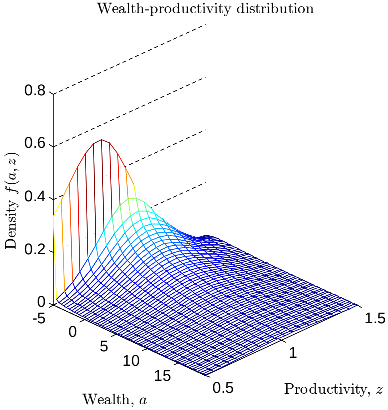
2. Matrix $\mathbf{A}$ is sparse so that $\left(\tilde{\rho}\mathbf{I} - \mathbf{A}^n\right)^{-1}$ can be computed efficiently

3. The distribution g is solved by finding the kernel of $\mathbf{A}^{\mathbf{T}}$

Results

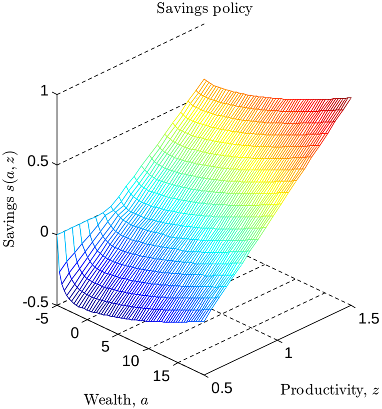
Competitive equilibrium





Results

Constrained-efficient solution



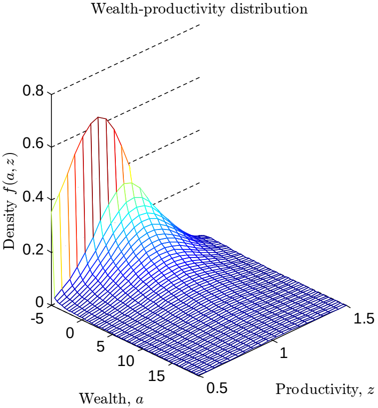


Table 1. Model results

	Competitive equilibrium	**Constrained optimum**
Aggregate capital, k	4.16	4.88
Output, y	1.67	1.77
Capital-output ratio, k/y	2.49	2.76
Interest rate (%), r	4.45	3.04
Pareto exponent , ζ	1.53	0.76

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Ongoing extensions of this paper

1. **Optimal transitional dynamics** in the current example (Aiyagari model)
2. Extension to "**jumps**" (Poisson shocks) $\longrightarrow$ straightforward
3. Other examples
    1. Misallocation due to financial frictions, ex. Moll (AER, 2014)
    2. Efficient allocation in job search models with learning, ex. Menzio and Shi (JPE, 2011) or Li and Weng (IER, forthcoming)

Related research

- **Games** between a major player and a distribution (Nuño and Thomas, 2016)
  - Optimal Monetary Policy in a Heterogeneous Monetary Union
  - Open-economy Bewley model with non-contingent long-term nominal debt
  - We analyze optimal inflation under commitment (Ramsey / open-loop Stackelberg) and discretion (Markov perfect Nash equilibrium)