Debt-Maturity Management with Liquidity Costs

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We document the presence of significant liquidity costs in Spanish sovereign debt auctions: the larger the auctioned amounts, the lower the issuance price relative to secondary-market prices. Motivated by this evidence, we characterize the optimal debt-maturity management problem of a government that issues finite-maturity bonds of various maturities, in the presence of such liquidity costs. This characterization allows us to quantify how the government's relative impatience, yield-curve riding, and expenditure smoothing shape the optimal debt-maturity distribution. The model can rationalize actual debt-management practices.

I. Introduction

Any government faces a large-stakes problem: to design a strategy for the quantity and maturity of its debt. This paper presents a new framework

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to think about and evaluate that design. We study the optimal debt-maturity management problem of a government subject to income and interest rate shocks. The framework makes two innovations. First, it puts forth liquidity costs as a central consideration: the notion that the larger a bond auction is, the lower the auction price. Second, it develops an approach to characterize the optimal debt program allowing for an arbitrary number of finite-life bonds. With this approach, we can analyze several forces that influence the optimal debt-management strategy.

Liquidity costs are a concern to practitioners, but to a large extent, their role in normative theoretical analysis has been absent. We start by estimating the relevance of liquidity costs in practice. To this end, we exploit a feature in the debt-issuance strategy of Spain. Spain regularly issues bonds in different maturity categories, namely, 3, 6, 9, 12, and 18 months and 3, 5, 10, 15, 30, and 50 years. Each bond in circulation has a set of characteristics, such as a coupon structure and maturity, and are identified by an International Securities Identification Number (ISIN). Instead of issuing different bonds in each auction, the Spanish Treasury often reissues identical bonds with the same ISIN as bonds that already circulate in the secondary market. For instance, the Treasury could issue in March 2017 a 5-year bond with the same ISIN as a 10-year bond issued in March 2012. This implies that when a new auction of the bond with the same ISIN happens, an identical bond is already trading in the secondary market. This allows us to compute the markup of market prices relative to auction prices. Because bonds share the same ISIN, their matching isolates any other potential legal or regulatory characteristics that could pollute the analysis.

To estimate liquidity costs, we collect data on the universe of Spanish Treasury auctions from January 4, 2002, to April 20, 2018. For each of the 2,579 auctions, we match identical-ISIN bonds to their secondary-market prices. Since most auctions are reissuances, we match about 80% of the total auctions of preexisting vintages with secondary-market prices. The average markup on the marginal price ranges from 1.4 to 5.7 bps (basis

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points) for bonds with maturity below 1 year and from 8 to 31 bps for longer-term bonds. With these markups, we estimate liquidity costs: the sensitivity of markups to auctioned amounts. We find evidence of considerable liquidity costs in bond markets, especially for bonds with maturity above 3 years.

To rationalize this evidence, we introduce a simple wholesale-retail model that produces liquidity costs. We build on Duffie, Gârleanu, and Pedersen (2005) and assume that the government auctions bonds to primary dealers, who then resell the bonds to their ultimate holders, investors. As in Kargar et al. (2021), primary dealers take time to liquidate their bond inventories. Since the market is over the counter (OTC), the greater the amount auctioned to dealers, the longer the resell time. Because dealers face high capital costs (e.g., in Bocola 2016), the longer the resell time, the costlier it is to redeploy the bonds. As a result, dealers bid at lower prices (relative to secondary-market prices) as the auction size at a given maturity increases. A final assumption is that bond markets are segmented by maturity, in the spirit of the preferred-habitat investors of Vayanos and Vila (2021). With these ingredients, the price sensitivity to auctioned amounts is summarized by maturity-dependent liquidity coefficients (or price impacts) equivalent to the ones measured in the data.

We study the optimal debt-management problem of a government that internalizes liquidity costs. In the presence of liquidity costs, the number and types of bonds that the government can issue are not innocuous. For that reason, we allow the government to issue an arbitrary number of finite-life bonds of different maturities, an exercise hitherto not carried out. Because of the curse of dimensionality, qualitative analysis is often relegated to highly stylized models, and quantitative models allow for only a small number of decaying perpetuities (as in Leland and Toft 1996).[1] In practice, governments simultaneously issue in multiple maturities. Furthermore, perpetuities are a rarity. Our analysis shows how the government's problem can be studied as if multiple artificial traders were in charge of issuing debt of a corresponding maturity. Each trader must apply a simple rule:

$$\frac{\text{issuance at maturity } \tau}{\text{GDP}} = \frac{1}{\text{liquidity coefficient at maturity } \tau} \times \text{value at maturity } \tau. \qquad (1)$$

[1] This limitation is easily understood. If we want to construct a yearly model where the government issues only 30-year bonds, we need at least 30 state variables, because a 30-year bond becomes a 29-year bond the following year, a 28-year bond the year after, and so on. By contrast, a bond that matures by 5% every year is still a bond that matures by 5% the year after its issuance.

This rule follows from a condition that equates marginal auction revenues to an internal debt valuation. The rule states that optimal issuances of a bond of maturity τ equal the ratio of a value gap to the liquidity coefficient of that maturity. The value gap is the proportional difference between the secondary-market price, computed using the international short-term rate, and the domestic valuation, computed using the government's discount rate. Unlike models where liquidity costs are absent, here the government's discount rate differs from the short-term rate. A positive value gap indicates the desire to arbitrage the difference between market prices and domestic valuations by the artificial traders. The liquidity coefficient modulates the willingness to arbitrage in a given maturity. As a result of this limited arbitrage, it is optimal to issue in all maturities at all times, as commonly done by treasuries. Since the domestic discount factor rate must be internally consistent at an optimum, a single equilibrium variable summarizes the problem, which features a continuum of control variables.[2]

The simple rule is convenient because it makes it possible to dissect the forces that shape the optimal debt-maturity profile through their impact on these rates. We characterize how these forces affect the level and weighted average maturity (WAM) of debt issuances through the elasticities of domestic valuations and bond prices, with respect to the parameters associated with each force.

In the steady state, the domestic rate depends only on the government's impatience relative to international investors'. We obtain an analytic expression for the optimal debt profile. We show that when liquidity costs are constant across maturity, the government should issue in all maturities, but with a pattern that increases with maturity. The government's greater impatience, compared to the market's, is key to the pattern of increasing in maturity. The intuition is as follows. Extending the maturity by 1 period delays the principal payment by 1 period. If the government were as patient as international investors, this delay would not bring additional benefit. However, since the government is more impatient, the delay brings a benefit because the government uses a higher discount rate than markets. Thus, the government always prefers to issue at long maturities. However, the government spreads out its issuances across maturities to mitigate liquidity costs. We show that, as relative impatience increases, issuances increase at all maturities but the average maturity falls. The benefit for the government of delaying a principal payment increases

[2] The domestic discount rate is the solution of a fixed-point problem: conjectured expenditure path maps to a domestic discount rate. This discount rate generates an issuance path via the optimal rule. Ultimately, the issuance path must be consistent with the expenditure path's debt service. The conjectured expenditure path must coincide with the actual expenditure path obtained by applying the issuance rule at the optimum.

with impatience, but more so at the lower end of the yield curve. Therefore, as impatience increases, the WAM decreases.

Understanding the role of impatience is key to understanding the forces that drive optimal debt issuances. Throughout a deterministic transition, two forces shape the dynamics of the value gap: expenditure smoothing and yield-curve riding. Expenditure smoothing is activated if the government has a positive intertemporal elasticity of substitution. Its desire to smooth expenditures induces a higher domestic discount when expenditures fall. Hence, smoothing acts as a temporary increase in impatience and induces greater overall borrowing, together with a shortening of debt maturity. For example, consider an economic recovery where the path of revenues is momentarily low. With liquidity costs, the government cannot smooth expenditures perfectly. As a result, domestic discounts are high throughout the recovery. High discounts increase the value gap, but more so for longer bonds because of the compounding of discount rates.

Yield-curve riding is activated when there are predictable changes to the yield curve. For example, if the government faces a temporary increase in short-term rates, the value gap narrows, especially for short-term debt. In this case, the optimal amount of debt falls, and the average maturity of issuances increases. In general, yield-curve riding is the strategy of altering the debt maturity against the direction of changes in the yield curve, something we observe in practice that has not been rationalized by models without liquidity costs. Naturally, if the government also cares about smoothing expenditures, changes in the yield curve carry effects through both yield-curve riding and expenditure smoothing.

We build on these results and compare the predictions of our model regarding liquidity costs with those obtained from the auction microdata. We employ the theoretical formula (1) linking issuances, bond prices, and liquidity costs to derive the liquidity costs consistent with the Spanish issuance profile across maturities over the period, and we compare them with the empirical results from the microdata. Both liquidity cost measures yield similar maturity profiles, with liquidity costs approximately zero for maturities below 18 months and increasing steeply for maturities above 10 years.

Then, we numerically solve the model and use it to analyze the optimal responses to small income and interest rate shocks. We find that the responses of the debt and the WAM are consistent with the correlations found in the Spanish data over the period 2002–18. In a final section of the paper, we discuss how other considerations, such as tax smoothing and stock effects, would interact with liquidity costs.

Related literature.—This paper makes a contribution in two areas. The first is the area of finance that studies liquidity frictions. The second area is the normative literature that studies the optimal management of public debt. With regard to the study of liquidity frictions, there is ample evidence of their presence in asset markets, as surveyed by Duffie (2010) or Vayanos and Wang (2013).[3] As we noted above, liquidity costs can emerge from OTC frictions, as in Duffie, Gârleanu, and Pedersen (2005). This area has recently received attention in light of recent disruptions in the US Treasury markets (see Duffie 2010; Kargar et al. 2021). Relative to this literature, this paper studies the optimal management of public debt in the presence of liquidity frictions. We also build on the literature on bond market segmentation, following Vayanos and Vila (2021). Namely, we study an environment where bonds of different maturity are issued in de facto segmented markets. As a result, the issuing government confronts a demand system for bonds that differ by maturity (see Koijen and Yogo 2019 and Gabaix and Koijen 2021 for asset-pricing models with demand systems).

Regarding the normative literature, guidelines for optimal debt management emerge from international and public finance. In international finance settings, national income is treated as exogenous and the government chooses a debt profile to maximize the net present value of utility from domestic consumption, on behalf of its citizens. In public finance settings, expenditures are exogenous and the government chooses a debt profile to minimize the net present value of tax distortions. Aside from this difference in structure, both areas base their prescriptions for optimal debt management on common economic forces: the impatience of the government relative to investors, the smoothing of objective functions, insurance across states, and incentives in environments where strategic decisions matter. The main contribution of this paper is to investigate, analytically and quantitatively, how liquidity costs interact with smoothing motives in shaping the optimal debt structure. Our model shares the structure of international finance models, but we also show that the model can be recast into a public finance formulation.

In public finance, Barro (1979) showed that, absent risk, a government should design its debt profile to smooth tax distortions, akin to consumption smoothing in international finance settings. In both public and international finance settings, smoothing has implications for the stock of debt, but not for maturity—because bond prices are arbitrage-free and the government's discount rate coincides with the short-term rate. Here, we show that liquidity costs break that relationship and open a value gap, as described above. In the presence of liquidity costs, it is ideal to shorten the debt maturity when the government desires to smooth a temporary decline in revenues. Furthermore, liquidity costs uncover

[3] See also Cammack (1991), Spindt and Stolz (1992), Duffee (1996), Fleming (2002), Green (2004), Fleming and Rosenberg (2008), Pasquariello and Vega (2009), Krishnamurthy and Vissing-Jorgensen (2012), Pelizzon et al. (2016), and Breedon (2018).

a yield-curve riding force, a force that is not present when the government's discount rate coincides with short-term rates. Yield-curve riding dictates that debt maturity should move in the opposite direction to the slope of the yield curve.

The literature has explored other motives for maturity management. One of them is insurance. In a stochastic extension of Barro (1979), Lucas and Stokey (1983) show that a government that accesses state-contingent debt should smooth taxes, state by state. In turn, Angeletos (2002) demonstrates that a government can implement the complete-markets optimal taxation in Lucas and Stokey (1983) by appropriately designing a portfolio of fixed-income bonds that generates capital gains that perfectly offset shocks to revenues. Departing from complete markets, Aiyagari et al. (2002) studies the Lucas and Stokey (1983) problem when the government can issue in a single maturity. Aiyagari et al. (2002) showed that, in that case, self-insurance induces lower debt levels.[4] Another is sovereign default. Bulow and Rogoff (1988) identified that long-term debt is prone to debt dilution, the idea that once the long-term debt is issued, the price of a new issuance does not internalize the increased default premia on past debt (see Aguiar et al. 2019 for a dynamic analysis of this phenomenon).[5] Our model does not capture these motives. In relation to this literature, the analysis in our paper can be thought of as describing the optimal debt dynamics after small shocks linearized around a deterministic steady state, in the spirit of dynamic stochastic general equilibrium models.

A paper close to ours is Faraglia et al. (2019), which also studies optimal maturity management with finite-life bonds and stresses the role of liquidity frictions.[6] That study calibrates a closed-economy model with recurrent shocks to match maturity and debt-level moments. Our paper

[4] In turn, Buera and Nicolini (2004) note that, with the observed volatility of bond prices in the data, governments would have to hold substantial debt positions to implement the Lucas and Stokey (1983) tax sequence. A particular case is studied in Barro (2003), where income is deterministic but the discount rate is stochastic. That paper shows that governments should issue perpetuities. A similar prescription emerges in international finance, where rollover risk of short-term debt may prompt defaults, an idea that goes back to Calvo (1988) and Cole and Kehoe (2000) and is studied in depth by Bocola and Dovis (2019). Lustig, Sleet, and Yeltekin (2008) analyze how, even in the presence of short-selling constraints, long-term debt provides a hedge against fiscal shocks.

[5] Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012, 2015) study quantitatively the positive and normative properties of a model in which the government borrows by issuing long-term bonds, a setup that is prone to debt dilution. Arellano and Ramanarayanan (2012) and Hatchondo, Martinez, and Sosa-Padilla (2016) study optimal debt-maturity management when the government has access to both short- and long-term debt.

[6] Valaitis and Villa (2022) present a machine-learning algorithm to solve debt-management models with a large number of bonds. Kiiashko (2022) studies the optimal debt-maturity structure in a tractable model with default.

and Faraglia et al. (2019) share a common message: liquidity frictions are essential to reconcile debt-management practices with model predictions. We complement that work along two dimensions. First, by focusing on a deterministic setting, we can solve for the steady-state distribution of debt analytically. This allows us to characterize how expenditure smoothing and yield-curve riding affect the level and maturity distribution of debt. Second, we provide a direct measure of liquidity costs by estimating them with data from public debt auctions. This estimation allows us to contrast the liquidity costs inferred by the model to match the average debt-maturity distribution against the liquidity costs from micro estimates. We show that the patterns and magnitudes of model-implied and estimated liquidity costs are similar.

Two other papers are also close to ours: Bhandari et al. (2017) and de Lannoy et al. (2022). Bhandari et al. (2017) shows that self-insurance affects optimal debt maturity in an open economy with distortionary taxes. That paper characterizes, under quasi-linear preferences, the asymptotic moments of debt and maturity in a two-bond environment. In that model, the maturity and debt levels have a positive correlation. In contrast to Bhandari et al. (2017), here, liquidity costs prescribe issuances of debt at all maturities, but the correlation with maturity depends on the relevant economic force that operates after a shock. De Lannoy et al. (2022) considers an open-economy environment in which a government issues in a large number of maturities and faces risk in revenues and interest rates. Importantly, as in our environment, debt issuances at different maturities have a price impact, and the sensitivity of that price impact is also a source of risk. The authors show that the optimal solution to the debt-management problem can be cast in terms of distance to a target portfolio that hedges risk à la Angeletos (2002). The authors then show that hedging interest rate risk is quantitatively the most important driver of the optimal debt portfolio.

II. Evidence of Liquidity Costs

In this section, we estimate the liquidity costs in Spanish sovereign debt issuances. To this end, we exploit an institutional characteristic that is common in sovereign debt markets: treasuries typically issue the same bonds over time. For instance, the treasury may issue 5-year bonds with expiration dates anywhere from 45 to 75 months. The advantage of this is that, at the time of each reissuance in the primary market, an identical bond is trading in the secondary market. We can thus compare prices in the primary and secondary markets during auction days to compute the markup charged by primary dealers, who intermediate between primary and secondary markets. We can then compute the sensitivity of these markups to auction volumes, a measure of liquidity costs.

A. *Structure of Issuances*

The Spanish Treasury issues debt on behalf of the central government. Bond issues are grouped into vintages identified by an ISIN. Each vintage is associated with the same expiration date and the same monthly coupon. After a new vintage is issued, the Treasury often auctions bonds corresponding to that vintage at later dates. Thanks to our conversations with practitioners, we understand that grouping issuances into common vintages to be issued repeatedly simplifies the legal and regulatory transaction costs through standardized legal parameters. This feature is critical for our estimation.

The Treasury catalogs bonds into three categories: *Letras del Tesoro* (henceforth, "bills") are zero-coupon bonds with approximate original maturities of 3, 6, 9, 12, or 18 months; *Bonos* and *Obligaciones del Estado* are constant-coupon bonds with approximate maturities of about 3 or 5 years (*Bonos*) or about 10, 15, 30, or 50 years (*Obligaciones del Estado*). The Treasury also issues inflation-indexed bonds, which we do not analyze, since their volume is small. We refer to the latter categories simply as "bonds."

We classify bonds into 11 maturity categories that roughly correspond to the Spanish Treasury's monthly categories: 3 (between 0 and 4.5), 6 (between 4.5 and 7.5), 9 (between 7.5 and 10.5), 12 (between 10.5 and 14.5), 18 (between 14.5 and 20.5), 36 (between 20.5 and 44.5), 60 (between 44.5 and 74.5), 120 (between 74.5 and 144.5), 180 (between 144.5 and 210.5), 360 (between 210.5 and 410.5), and 600 (more than 410.5) months. We define the collection of categories as $\mathcal{M}$ and use it in the rest of the paper.

B. *Auctions*

The Spanish Treasury issues debt at competitive auctions. Auction participants include primary dealers and institutional investors. Auctions occur weekly, on two Tuesdays for bills and two Thursdays for bonds. The Treasury publishes tentative auction schedules several months in advance, where it announces the size of each auction corresponding to each vintage.

Auctions are sequential: the first round (the competitive round) is open to any market participant, whereas the second round (the noncompetitive round) is reserved for primary dealers, registered frequent participants. We focus only on the first round, which accounts for the bulk of issuances. Auctions are modified Dutch auctions: competitive bidders submit sealed bids of price-quantity pairs. Noncompetitive bids specify only a quantity. The allotment works as follows. All noncompetitive bids are accepted, by default. Competitive bids are sorted by price. The Treasury's offered amount, net of noncompetitive bids, determines a stop-out marginal price. This marginal price is such that all competitive bids at or above the marginal price are accepted. As a result of this structure, the sum of competitive and noncompetitive accepted bids equals the auctioned amount. The allocation price is calculated in two tiers: bids falling between the marginal and rounded-up weighted average prices pay their corresponding bid price. Bids above the weighted average price and noncompetitive bids pay the weighted average price.[7] Securities are issued 3 working days after the auction—the following Friday for bills and Tuesday for bonds.

C. *Data*

The Spanish Treasury provided us with the universe of auctions from January 4, 2002, to April 20, 2018, excluding inflation-indexed instruments. Adding across auctions, we construct a time series for the Spanish debt stock, accurately approximating the stock reported in the national accounts. The data cover almost the entire universe of debt issuances. For each of the 2,579 auctions, the panel includes the ISIN code, the coupon, the issuance date, the inception date of the vintage corresponding to the bond issued, the expiration date, the total tender amounts received in both the first (competitive) round and the second (noncompetitive) round, the total tender amounts allocated, and the marginal and average prices.

Figure 1 displays scatter plots where each dot represents an auction with the date on the x-axis and maturity on the y-axis. Different panels group auctions by their category. We can observe that the Treasury issues in each maturity almost every month. There are only a few gaps in the 9- and 18-month bills and the 15-year bonds; 50-year debt is special, in that it was introduced later in the sample. We also observe a saw-like pattern in all categories. This pattern reflects that sequential issuances in a given vintage have approximately the same maturity. Take, for example, the 3-year category. Typically, the first auction of a 3-year vintage has a maturity of slightly above 36 months. After the first issuance, bonds of the same vintage are reissued every following month for about a semester. Hence, by the time the vintage reaches its last auction, the issued bond has a maturity of 30 months. Past the last auction, a new vintage is reopened, starting again from a maturity above 36 months. The same is true of other categories. Thus, although categories do not exactly correspond to a specific maturity, we observe an almost seasonal pattern centered around

[7] A cap on the marginal price is placed to protect some investors from overpaying. The segmentation of bids into competitive and noncompetitive is done to protect the auction from collusion, by placing large and frequent investors in the noncompetitive segment.

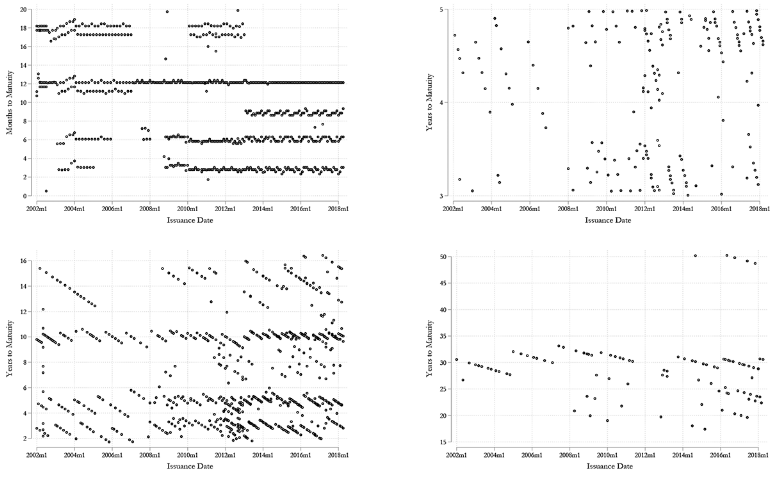

FIG. 1.—Issuance pattern. Each point in each plot corresponds to an auction. The vertical axis denotes the issued security's maturity at a given date.

the category's maturity—the pattern is not exactly seasonal because the length of the vintages is not uniform.

These observations guide our theory. First, these observations call for a theory where economic forces induce continuous issuances in several maturities instead of issuances concentrated in a single maturity. Second, the pattern justifies bundling bonds into specific maturity categories—from the Treasury's perspective, 40- and 36-month bonds are approximately the same bonds.

Using the ISIN code, we merge these data with secondary-bond-market prices obtained from Bloomberg. For each ISIN traded on a given day, the data include the corresponding first and last bid and ask prices. Only if the vintage has already been opened can we match auction prices with secondary-market prices. Since the matched prices are by ISIN, we match identical-maturity securities, isolating any other potential legal or regulatory characteristics that could pollute the analysis. Since most auctions are reissuances (85%), we match about 80% of the total auctions of preexisting vintages (2,077) with a secondary-market price. We normalize the issuance amount by the 18-month moving average of the month's nominal GDP to account for changes in both the economy's size and the price level.[8] We denote the issuance as a percentage of monthly GDP

[8] We also match each auction with the nominal GDP of the corresponding month. We obtain nominal monthly GDP from a dynamic factor model, based on Camacho and Perez-Quiros (2009), that computes short-term forecasts of Spanish GDP growth in real time.

at a given date by $I_t(\tau)$. We use an 18-month moving average to avoid confounding issuance-flow variation in the denominator.

D. Markups

We measure the auction markup as the normalized difference between the auction price and the market price. For each issuance of maturity τ at date t, we construct the markup, using the formula

$$\text{Markup}_t(\tau) \equiv \frac{\psi_t(\tau) - q_t(\tau)}{\psi_t(\tau)}, \tag{2}$$

where $q_t(\tau)$ is the auction price and $\psi_t(\tau)$ represents the secondary-market price of a bond. We express markups in basis points. The market price is computed as the average of the bid and the ask at the end of the auction's date. A positive ratio indicates that the average participant earned a markup. Otherwise, the participant overpaid. There are multiple prices in each auction, corresponding to different bids: the marginal price, the average price, and the weighted-average-above-the-marginal (WAAM) price. We can construct markups for each auction price measure.

Table 1 reports the summary statistics for markups and issuances over GDP. The average markup on the marginal price ranges from 1.4 to 5.7 bps for bills. Among bonds, average markups range from 8 to 31 bps. We also report markups constructed with the WAAM and the average price, which are smaller across all categories because they are constructed from higher bids. Using the WAAM, for instance, 1-year bonds have the highest markup (5 bps) among bills. Among bonds, markups are typically negative. For example, markups are −33, −42, and −36 bps for the 10-, 15-, and 30-year bonds, respectively. In the analysis, we use the WAAM markup.[9] The substantial markups in primary auctions suggest the presence of liquidity frictions.

E. Liquidity Costs

After constructing markups, we estimate liquidity costs as a function of issuances. using

$$\text{Markup}_t(\tau) = \alpha(\tau) + \beta_t + \Lambda(\tau) \cdot I_t(\tau) + \epsilon_t(\tau), \tag{3}$$

[9] The regression estimates for marginal-price markups are insignificant. We interpret this finding as suggesting that the distribution of bids, which we do not observe, is concentrated around the marginal price. If this is the case, the marginal price is not sensitive to the auction size. Hence, auction-size effects are captured by the WAAM markup, which motivates its use.

where t and $\tau \in \mathcal{M}$ are the date and maturity category of the auction, respectively. The term $\alpha(\tau)$ is a maturity fixed effect, β_t is a month fixed effect, $I_t(\tau)$ are monthly issuances over yearly GDP, and $\epsilon_t(\tau)$ is the error term. The *price impact* coefficient, $\Lambda(\tau)$, measures the sensitivity of markups to issuances; $\Lambda(\tau)$ is estimated for each maturity group—except for the 50-year bonds, because of lack of observations.

The estimates of $\Lambda(\tau)$ are presented in figure 2 and table D1. The x-axis of the figure corresponds to a maturity category and the y-axis to the coefficients, expressed in basis points, of markup per percentage of monthly issuances over annual GDP. Each diamond marks a point estimate, and the bars indicate the 1%, 5%, and 10% confidence intervals. We cannot find evidence of liquidity costs for the maturities to the left of the dashed line (bills). For those maturities to the right of the dashed line (bonds), the liquidity coefficient $\Lambda(\tau)$ is positive and significant at the 1% confidence level. The estimated values of liquidity costs among bonds are not negligible. The coefficient values range from a maximum of 56 bps for the 30-year bonds to 8 bps for 3-year bonds. These results are in line with the empirical literature, which has documented the presence of liquidity costs in bond markets. For example, see Madhavan and Smidt (1991), Madhavan and Sofianos (1998), Naik and Yadav (2003), or Hendershott and Seasholes (2007).

These results support the notion that liquidity costs are an important consideration for optimal debt issuances. As a robustness test, table D2 reports the regression estimates of $\Lambda(\tau)$, using a parametric form that is linear in maturity. Both tables D1 and D2 also present estimates splitting the sample into subperiods. Estimate magnitudes are similar and stable across specifications and subsamples, so we are confident that outlier auctions do not drive our results.

Considering that Spain issues, on average, about 17% of GDP per year, adequately managing its bond issuances is important for fiscal outcomes. To get a sense of scale, consider the following thought experiment. Suppose that Spain has to roll over its debt but can choose to spread issuances over 12 months or reissue them all in a single auction. Applying a linear pricing formula, the relative loss on revenues from concentrating the issuances is $\Lambda(\tau) - 12 \cdot \Lambda(\tau)$. With a rough estimate of the liquidity cost $\Lambda(\tau)$ of 10 bps, the loss in revenue is about 1.1% of the total issuances. Spain would save up to 0.2% of its GDP, around \$2.5 billion per year, by designing an optimal strategy.

To conclude the empirical analysis, we discuss several features of the data and the estimation. First, we find that many auctions show negative markups, which may seem counterintuitive in light of the theory that we develop in the rest of the paper. In practice, auctions involve multiple dealers who differ in capital costs, access to the secondary market, and risk profile and, ultimately, can make mistakes. This is reflected in the

TABLE 1
Descriptive Statistics

	Months										
	3	6	9	12	18	36	60	120	180	360	600
	Issuances over GDP (%)										
Mean	.84	.92	1.93	2.11	1.05	1.38	1.37	1.60	1.13	.92	1.14
SD	.76	.93	1.23	1.85	.85	1.16	1.15	1.85	1.34	1.15	1.02
Minimum	.00	.00	.02	.00	.01	.00	.01	.00	.01	.00	.05
p50	.68	.70	2.30	1.23	1.09	1.19	1.13	1.22	.73	.74	1.18
Maximum	4.26	5.34	4.15	5.89	3.38	5.04	5.76	11.70	8.11	8.00	3.23
Observations	189	222	86	301	130	254	295	332	123	139	8
	Markup on Marginal Price (Basis points)										
Mean	1.42	3.75	1.24	5.72	5.39	7.64	11.72	6.31	6.96	30.73	. . .
SD	3.19	8.93	3.22	14.89	42.92	66.32	84.42	101.25	100.58	158.69	. . .
Minimum	−3.91	−4.85	−6.39	−16.89	−254.30	−300.36	−409.73	−375.07	−301.40	−493.36	. . .
p50	.77	1.47	.13	2.32	3.81	4.52	15.28	18.26	13.70	28.34	. . .
Maximum	23.08	71.64	11.50	119.09	162.67	367.80	433.98	341.91	332.16	716.15	. . .
Observations	112	125	44	170	87	151	161	167	62	69	0

	Markup on Weighted Average Above Marginal (WAAM) Price (Basis points)										
Mean	1.05	2.86	.09	4.51	2.04	−18.84	−22.36	−32.82	−42.02	−36.46	73.04
SD	3.26	7.73	3.26	15.38	51.67	84.507	106.35	117.63	131.83	174.83	219.75
Minimum	−7.55	−12.03	−17.87	−25.79	−306.65	−414.68	−596.55	−467.97	−437.35	−513.13	−207.78
p50	.35	.95	−.3	1.53	3.30	−2.87	−6.67	−19.21	−22.16	−27.54	62.25
Maximum	23.09	67.93	9.99	119.09	162.67	347.73	427.35	328.52	302.31	676.14	411.08
Observations	189	220	86	301	130	254	295	332	123	139	8
	Markup on Average Price (Basis points)										
Mean	.86	2.75	.42	3.43	.73	−.465	.43	−10.91	−10.34	14.64	. . .
SD	2.91	8.38	2.96	14.55	43.31	67.30	88.30	107.91	102.10	164.02	. . .
Minimum	−4.76	−11.42	−7.85	−25.27	−265.13	−315.53	−445.58	−430.75	−327.61	−512.18	. . .
p50	−.32	.86	−.26	1.26	1.35	1.38	1.48	.40	4.91	13.82	. . .
Maximum	19.70	69.17	10.10	107.47	144.49	348.27	430.44	333.24	306.29	678.23	. . .
Observations	110	122	44	165	83	145	154	153	58	64	0

Note.—The descriptive statistics correspond to the full sample. Issuances over GDP are computed as the total issuance at the auction of date $t \in \mathcal{M}$ divided by the monthly GDP at the auction month. The average price is constructed as the weighted average between the marginal and the WAAM, where the weights are given by the fraction assigned to bids above and below the marginal. “Observations” corresponds to the total number of auctions for each of the maturity categories. p50 = 50th percentile.

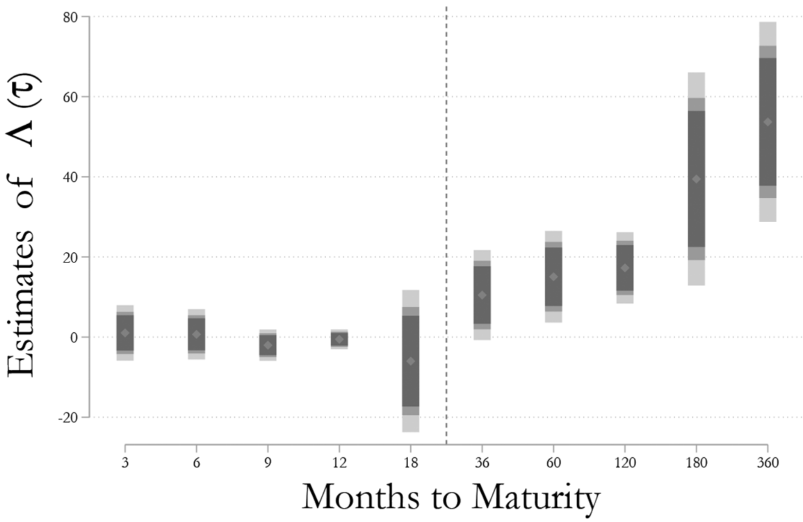

FIG. 2.—Regression coefficients of price impact, $\Lambda(\tau)$. Each bar corresponds to an estimate of $\Lambda(\tau)$ in basis points over monthly issuances (expressed as a percentage of annual GDP). Diamonds represent the point estimates, and the bars correspond to the 1%, 5%, and 10% confidence intervals.

significant dispersion of markup estimates, including the negative values. It should not, however, blur the conclusion that, on average, the government faces a downward-sloping demand curve for its debt issuances. Second, we should note that our approach may underestimate the price impact of auctions if secondary-market prices fall in the days that precede auctions, as found in Lou, Yan, and Zhang (2013). Third, as we noted above, the estimate of the price impact $\Lambda(\tau)$ is not statistically significant for bonds of short maturity (bills). This lack of significance should not come as a surprise: estimating price impacts at short maturities requires substantially more data than we have. This is because short-term bonds feature a price very close to their face value. To see this point, consider a bond of 1-day maturity with a face value of 100: its price will be extremely close to 100. Thus, there is little room for price variation and, consequently, the lack of significance at the short end of the yield curve should be expected. Another reason why we may not find evidence of a price impact on bills is that their clientele is different from that of bonds. According to practitioners at the Spanish Treasury, holders of bills are typically dealers who use bills to manage liquidity, holding them to maturity. This contrasts with the intermediary role of primary dealers in bonds of longer maturity, a central feature of the theory that we lay out in the next section.

III. Debt Management with Liquidity Costs

In this section, we present a wholesale-retail model of the bond market with OTC search frictions that produce liquidity costs. We then integrate these liquidity costs into the problem of a government that issues debt at different maturities and obtain the optimal debt-management strategy. All proofs are in the appendix.

A. OTC Model of Liquidity Costs

We consider an auctioned quantity ι of bonds of specific characteristics (maturity, coupon structure) at a given date t. A continuum of risk-neutral primary dealers purchase bonds in this auction (the primary market). Dealers then resell bonds to international investors, the ultimate bond holders, in a secondary market. International investors discount cash flows at a risk-free interest rate, r_t^*. Following Duffie, Gârleanu, and Pedersen (2005), dealers have a higher capital cost than investors, $r_t^* + \eta$, where $\eta > 0$ is an exogenous premium.[10] After the auction, investors contact dealers at a constant rate. The contact flow is $\mu \cdot y_{ss}$ per instant, where μ is a customer flow parameter and y_{ss} a measure of economic activity. Each contact results in a bond purchase by investors. Thus, within an interval Δt, the bonds sold by the dealer are $\mu \cdot y_{ss} \cdot \Delta t$. Hence, dealers will take time to liquidate the bond inventory ι. Critically, the larger the auction, the longer the resell time. Together with dealers' higher capital cost, the larger the auction, the lower the auction price.[11] Next, we derive the expression for this liquidity cost.

1. Secondary-Market Prices

A given auction offers a quantity ι of bonds with identical structure: each bond matures in τ time, has a normalized principal of one good, and pays an instantaneous coupon δ. The secondary-market price of the bond, ψ, is given by an arbitrage-free condition:

$$\psi_t(\tau) = e^{-\int_t^{t+\tau} r_u^* du} + \delta \int_t^{t+\tau} e^{-\int_t^s r_u^* du} ds. \tag{4}$$

In turn, $\psi_t(\tau)$ can be expressed as the solution to a partial differential equation (PDE),

[10] One way to interpret η is as limited intermediary risk-bearing capacity (along the lines of Bocola 2016).

[11] Previous papers have also explored inventory management of financial intermediaries as a source of liquidity costs. See, for instance, Ho and Stoll (1983), Grossman and Miller (1988), Huang and Stoll (1997), or Weill (2007).

$$r_t^* \psi_t(\tau) = \delta + \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial \tau}, \quad (5)$$

with boundary condition $\psi_t(0) = 1$. The first term of this PDE is the coupon flow, the second term is the capital gain, and the last term captures how the bond's maturity reduces with time.[12] The government repays the principal at the date of maturity, $\tau = 0$.

2. Bond Inventories and Primary-Market Valuations

Now assume that, at time t, a dealer wins the auction and buys the entire inventory ι. The bond inventory that remains with the investor by time s after the auction is $\max(\iota(\tau) - \mu y_{ss} \cdot s, 0)$. Hence, the bond inventory is exhausted by time $\bar{s}(\iota) = \iota/(\mu y_{ss})$ after the auction. The intensity of bond sales at the instant s therefore is

$$\gamma(s; \iota) = \frac{1}{\bar{s}(\iota) - s} \quad \text{for } s \in [0, \min\{\tau, \bar{s}(\iota)\}).$$

We treat $\gamma(s)$ as a Poisson intensity at which dealers sell an individual bond.[13]

To obtain the dealer's valuation of the bond, we assume that dealers extract all the surplus from international investors. The dealer's valuation is $q_{t+s}(\tau)$. The dealer's valuation satisfies

$$(r_{t+s}^* + \eta) q_{t+s}(\tau - s) = \delta + \frac{\partial q}{\partial t} - \frac{\partial q}{\partial \tau} + \gamma(s)(\psi_{t+s}(\tau - s) - q_{t+s}(\tau - s)). \quad (6)$$

The expression is similar to equation (5), but there are two differences. First, the discount rate for the dealer includes the higher cost of capital. Second, q captures the resale value of the dealer. When the dealer is contacted by an investor, which occurs with endogenous intensity $\gamma(s)$, the dealer's valuation jumps from the internal value q to the secondary price ψ because the dealer extracts all the surplus.

We assume that auctions are competitive. At the date of the auction, $s = 0$, the dealers bid $q_t(\tau)\iota$. The valuation $q_t(\tau)$ is a function of ι, which can be observed through the dependence of γ on ι. Appendix A1 presents the exact formula and a first-order linear approximation for $q_t(\tau)$ around small issuances. The approximation is

[12] Appendix E includes the equivalence between PDE and integral formulations for all the equations in the paper.

[13] The customer flow is $\mu \cdot y_{ss}$ per instant of time Δ. By time s after the auction, the remaining inventory is $\iota - \mu y_{ss} s$. Thus, the chance that a given bond is sold in an interval Δ is $\mu \cdot y_{ss}\Delta/(\iota - \mu y_{ss} s)$. Dividing by Δ and rearranging terms yields the expression for $\gamma(s)$.

$$q_t(\tau,\iota) = \underbrace{\psi_t(\tau)}_{\text{market price}} - \underbrace{\frac{1}{2}\bar{\lambda}\psi_t(\tau)\iota}_{\text{liquidity costs}}, \tag{7}$$

where $\bar{\lambda} = \eta/(\mu \cdot y_{ss})$ is the price-impact coefficient.

In equation (7), we interpret $(1/2)\bar{\lambda}\psi_t(\tau)\iota$ as liquidity costs. These costs increase linearly with the holding cost (the spread η) and inversely with the contact flow μ (which reduces the holding time). In the rest of the paper, we study the problem of a government that internalizes the price impact in equation (7). We make no further references to the secondary OTC market.

B. *Debt-Management Problem*

The government confronts two exogenous deterministic processes, $\{y_t, r_t^*\}_{t\geq 0}$, that represent paths for income revenues and short-term risk-free interest rates, respectively. We employ the "ss" subscript to denote a variable's steady-state value.

There is a single, freely traded good. The government has preferences over expenditure paths, $\{c_t\}_{t\geq 0}$, given by

$$\mathcal{V}_0 = \int_0^\infty e^{-\rho t} U(c_t)dt,$$

where $\rho \in (0,1)$ is a discount rate and the instantaneous utility, $U(\cdot)$, is increasing and concave. We assume that $r_{ss}^* < \rho$.

The government issues bonds that differ by maturity, τ. The government issues a continuum of maturities, $\tau \in [0, T]$, where T is a maximum exogenous maturity. For ease of exposition, unless stated otherwise, we assume that $\delta = r_{ss}^*$, so that bonds trade at par in steady state. The government sells each bond in the auctions introduced above. We treat each maturity as a separate market.

The outstanding bonds with maturity τ at date t are $f_t(\tau)$, which we refer to as the "debt profile." The debt profile evolves according to the PDE

$$\frac{\partial f}{\partial t} = \iota_t(\tau) + \frac{\partial f}{\partial \tau}, \tag{8}$$

with boundary condition $f_t(T) = 0$ and $f_0(\tau)$ given. This PDE captures that, given τ and t, the change in the quantity of bonds of maturity τ, $\partial f/\partial t$, equals the issuances at that maturity, $\iota_t(\tau)$, plus the net flow of bonds, $\partial f/\partial \tau$.[14] The latter term captures the maturing of bonds. The government's budget constraint is

[14] When negative, issuances represent purchases.

$$c_t = y_t - f_t(0) + \int_0^T [q_t(\tau, \iota)\iota_t(\tau) - \delta f_t(\tau)]d\tau, \tag{9}$$

where y_t is the (nonfinancial) government's revenue, $-f_t(0) - \delta\int_0^T f\, d\tau$ are the coupon principal and coupon payments, and $\int_0^T q\iota\, d\tau$ is the sum of all bond auction revenues. The government solves

$$V[f_0(\cdot)] = \max_{\{\iota_t(\cdot), f_t(\cdot), c_t\}_{t\geq 0, \tau\in[0,T]}} \int_0^\infty e^{-\rho t} U(c_t)dt, \tag{10}$$

subject to equations (8) and (9), the initial condition f_0, and internalizing the auction-price impact effect of $\iota_t(\tau)$ as given by equation (7).

C. *Optimal Debt Management*

We note that $V[f_0(\cdot)]$ is not a value function but a value functional, as it maps a debt profile $f_0(\cdot)$ into $\mathbb{R}$. In order to solve the government's dynamic programming problem, we employ variational techniques. We formulate the Lagrangian of equation (10):

$$\mathcal{L}[\iota, f] = \int_0^\infty e^{-\rho t} U\left(y_t - f_t(0) + \int_0^T [q(t, \tau, \iota)\iota_t(\tau) - \delta f_t(\tau)]d\tau\right)dt$$
$$+ \int_0^\infty \int_0^T e^{-\rho t} j_t(\tau)\left(-\frac{\partial f}{\partial t} + \iota_t(\tau) + \frac{\partial f}{\partial \tau}\right)d\tau\, dt,$$

where we substituted out expenditures from the budget constraint (eq. [9]). The second line attaches a Lagrange multiplier $j_t(\tau)$ corresponding to each constraint imposed by equation (8). A variational argument implies that no variations around the optimal issuance, ι, can improve the Lagrangian. This leads to the following first-order condition:

$$U'(c_t)\left(q(t, \tau, \iota) + \frac{\partial q}{\partial \iota}\iota_t(\tau)\right) = -j_t(\tau). \tag{11}$$

Furthermore, the argument also implies that no infinitesimal variation around f can improve the Lagrangian. This requires j to satisfy

$$\rho j_t(\tau) = -U'(c_t)\delta + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau}, \quad \tau \in (0, T], \tag{12}$$

with boundary and transversality conditions $j_t(0) = -U'(c_t)$ and $\lim_{t\to\infty} e^{-\rho t} j_t(\tau) = 0$. Relative to standard optimal control, where Lagrangians are connected through time given by an ordinary differential equation, here Lagrangians are connected through time and maturity through a PDE.

We translate j from utils into good units through $v_t(\tau) \equiv -j_t(\tau)/U'(c_t)$. We refer to v as the "domestic valuation" of the (τ, t) bond. We reexpress equations (11) and (12) as

$$\underbrace{\frac{\partial q}{\partial \iota}\iota_t(\tau) + q(t,\tau,\iota)}_{\text{marginal revenue}} = \underbrace{v_t(\tau),}_{\text{domestic valuation}} \tag{13}$$

and

$$r_t v_t(\tau) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \quad \tau \in (0, T] \text{ and } v_t(0) = 1, \tag{14}$$

where

$$r_t \equiv \rho - \frac{U''(c_t)c_t}{U'(c_t)}\frac{\dot{c}_t}{c_t}. \tag{15}$$

The transformed necessary condition, equation (13), states that the optimal issuance of the (τ, t) bond must equate the marginal auction revenue to its marginal cost. The marginal revenue is the price per issuance, q, plus the price impact, $(\partial q/\partial \iota)\iota$. The marginal cost is encoded in the forward-looking valuation, v_t. The valuation satisfies equation (14) and thus shares a remarkable connection with the market-price equation (5). Both the domestic valuation and the price are net present formulas for the payment flows of each bond. The only difference is the applied discount rate—market prices use r_t^*, whereas domestic valuation uses an endogenous domestic discount rate, r_t. Since v satisfies the same PDE as ψ in equation (4), its integral solution must also be the same—after we replace r_t^* with r_t:

$$v_t(\tau) = e^{-\int_t^{t+\tau} r_u\,du} + \delta\int_t^{t+\tau} e^{-\int_t^s r_u\,du}\,ds. \tag{16}$$

The following proposition summarizes the results so far.

Proposition 1 (Optimal issuances). If a solution $\{c_t, \iota_t(\cdot), f_t(\cdot)\}_{t\geq 0}$ to equation (10) exists, then domestic valuations $v_t(\tau)$ satisfy the PDE (eq. [14]) and the optimal issuances $\iota_t(\tau)$ satisfy the issuance rule (eq. [13]). The evolution of the debt profile can be recovered from the law of motion for debt (eq. [8]), given the initial condition f_0. Finally, c_t and r_t must be consistent with the budget constraints, equations (9) and (15).

Whereas the optimal-issuance rule (eq. [13]) is valid for any generic price-impact function, we employ the approximate liquidity cost function, equation (7). Under this approximation, the optimal-issuance condition (eq. [13]) simplifies to

$$\iota_t(\tau) = \frac{1}{\bar{\bar{\lambda}}} \cdot \underbrace{\frac{\psi_t(\tau) - v_t(\tau)}{\psi_t(\tau)}}_{\text{value gap}}. \tag{17}$$

This rule states that the optimal issuance of a (τ, t) bond is the product of the value gap scaled by the inverse of the price impact. When the value gap is positive, the government should issue at that maturity, because the market price exceeds its cost assessment, summarized by its valuation. The force that limits that desire to arbitrage is liquidity, a force encoded in $1/\bar{\lambda}$.

Generically, valuations and prices differ, $\psi_t(\tau) \neq v_t(\tau)$, leading to issuances in all maturities, $\iota_t(\tau) \neq 0$. Intuitively, if the government had to issue in a single maturity, it would select the maturity with the largest value gap. With liquidity costs, the issuance rule dictates that the marginal auction revenue should equal the domestic valuation. This is a form of monopoly pricing. Since marginal revenues decrease with issuance size, the government spreads issuances along all maturities. A virtue of this model is that it rationalizes why countries issue simultaneously in several maturities.

We define the WAM of issuances,

$$\mu_t \equiv \frac{\int_0^T \tau \iota_t(\tau) d\tau}{\int_0^T \iota_t(z) dz},$$

a metric of average maturity. We are interested in the comparative statics about the WAM with respect to parameters that we associate with different economic forces. Let θ be a parameter of interest. The elasticity of the WAM with respect to θ, $\epsilon^{\mu}_{t,\theta} \equiv (\partial\mu_t/\partial\theta) \cdot (\theta/\mu_t)$, is related to the elasticity of the issuance at a given maturity, $\epsilon^{\tau}_{t,\theta} \equiv (\partial\iota_t(\tau)/\partial\theta) \cdot (\theta/\iota_t(\tau))$, as established in the following lemma.

LEMMA 1. (Monotone comparative statics). Assume that all issuances are positive, $\iota_t(\tau) > 0$. If the issuance elasticities, $\epsilon^{\tau}_{t,\theta}$ are increasing (decreasing) in maturity τ, $d\epsilon^{\tau}_{t,\theta}/d\tau > 0$ for all $\tau \in [0, T]$, then the elasticity of the WAM with respect to θ, $\epsilon^{\mu}_{t,\theta}$, is positive (negative).

This lemma provides a sufficient condition to obtain a monotone comparative static for the WAM. Since issuance elasticities depend only on valuations and prices, all we need is to test whether the elasticities of valuations and prices are monotone in maturity τ to understand the effects on maturities. We employ lemma 1 below.

D. Analysis

Proposition 1 holds for any arbitrary function U. To sharpen the predictions, for the rest of the paper we assume constant relative risk aversion utility, $U(c) = (c^{1-\sigma} - 1)/(1 - \sigma)$, where σ is the inverse intertemporal elasticity of substitution (IES). With these preferences, $r_t = \rho + \sigma \dot{c}_t / c_t$.

1. Frictionless Benchmark

Liquidity costs are central to our analysis, but it is useful to first describe the model's prediction in a benchmark case where liquidity considerations are absent, the case where $\bar{\lambda} = 0$. In this case, the solution to the government problem coincides with the solution to a standard consumption-savings problem with a single instantaneous bond, B_t, that evolves according to $\dot{B}_t = r^*{}_t B_t - y_t + c_t$. Any debt profile $f_t(\tau)$ is a solution to the original problem, provided that

$$B_t = \int_0^T \psi_t(\tau) f_t(\tau) d\tau, \qquad \forall\ t.$$

Under this benchmark case, the frictionless equation (13) holds trivially as r_t^* and r_t are equalized. Given that the yield curve is arbitrage free and the discount factor coincides with the interest rate, there is no way for the government to restructure debt to reduce its servicing cost. Moreover, all bonds are redundant, although the path of consumption is consistent with $r_t^* = r_t$ and an intertemporal budget constraint. Hence, the optimal maturity structure is undetermined, as noted by Barro (1979). Our next task is to describe how things change when the government faces liquidity costs.

2. The Long-Run Pattern of the Optimal Debt Profile

The asymptotic behavior of the solution as time goes to infinity can be characterized analytically; see appendix A5. There are two relevant cases that depend on the liquidity coefficient, $\bar{\lambda}$.

(i) *Low liquidity costs.*—If the price impact, $\bar{\lambda}$, is lower than a threshold $\bar{\lambda}_o$—expressed in terms of the model parameters—there is no steady state. Expenditures decrease asymptotically at the exponential rate $r_{ss}^* - \rho$, and the domestic discount rate r_t converges to a limit value $r_\infty(\bar{\lambda})$. This limit domestic discount rate is increasing and continuous in $\bar{\lambda}$, with bounds $r_\infty(\bar{\lambda}_o) = \rho$ and $r_\infty(0) = r_{ss}^*$. In the limit as the price impact converges to zero, $\bar{\lambda} \to 0$, the asymptotic distribution of debt issuances is

$$\lim_{\bar{\lambda}\to 0}\iota_\infty(\tau) = \frac{1 + \left[-1 + (r^*/\delta - 1)r_{ss}^*\tau\right]e^{-r_{ss}^*\tau}}{1 + \left[-1 + (r^*/\delta - 1)r_{ss}^* T\right]e^{-r_{ss}^* T}}\frac{\psi_{ss}(T)}{\psi_{ss}(\tau)}\Xi,$$

where $\Xi > 0$ is a constant that guarantees zero expenditures. This expression shows that, even as liquidity costs are made arbitrarily small, the asymptotic debt profile is determined.

(ii) *High liquidity costs.*—If the price impact is above the threshold, $\bar{\lambda} \geq \bar{\lambda}_o$, a steady state with positive expenditures exists. In this case, the domestic discount factor is ρ and the issuances are solved analytically:

$$\iota_{ss}(\tau) = \frac{1}{\bar{\lambda}}\left[1 - \frac{e^{-\rho\tau} + (\delta/\rho)(1 - e^{-\rho\tau})}{e^{-r_{ss}^*\tau} + \left(\delta/r_{ss}^*\right)\left(1 - e^{-r_{ss}^*\tau}\right)}\right], \tag{18}$$

which is positive, given that $r_{ss}^* < \rho$.

We further characterize the issuance distribution for sufficiently large liquidity costs.

PROPOSITION 2 (Issuances increase with maturity). Let $\bar{\lambda} > \bar{\lambda}_o$ and $\delta = r_{ss}^*$. Then, steady-state issuances increase with maturity, $\partial\iota_{ss}/\partial\tau > 0$.

To explain the increasing pattern, note that $\partial\iota_{ss}/\partial\tau = -(1/\bar{\lambda})(\partial/\partial\tau)v_{ss}(\tau)$. If we set $r_t = \rho$ in equation (16), we obtain

$$\frac{\partial v_{ss}}{\partial\tau} = -(\rho - \delta)e^{-\tau\rho} < 0, \tag{19}$$

a negative derivative, since $\rho > \delta = r_{ss}^*$. The logic behind the increasing pattern is simple: an increase in maturity delays the principal repayment by one instant. This delay is discounted by the government at rate ρ. The delay costs an additional coupon δ. This trade-off is evaluated τ periods ahead; hence, the trade-off is scaled by $e^{-\tau\rho}$.[15] Recall that in order to equalize marginal revenues to marginal valuations, the government must spread out issuances across all maturities. Since the valuation of longer-term debt decreases with maturity, the government accepts a lower marginal revenue on longer maturities, and this comes about with greater issuances at longer maturities. All in all, we obtain a pattern of issuances in all maturities, but one that is increasing in maturity.

It is important to clarify that the frequency of rollover is not the reason behind the government's willingness to issue greater amounts of long-term debt. The rollover frequency would be a consideration in an environment where, at an initial date, the government has to choose to issue

[15] If the bond is not issued at par, we must take into account the reduction in the bond price, $\partial\psi_{ss}/\partial\tau$, in the issuance decision, but the same logic follows.

permanently in a given maturity, τ. In the problem studied here, the envelope theorem guarantees that when considering the issuances at different maturities at any point in time, the government compares only the payment flows associated with each bond, disregarding the maturity of the bonds it will use to refinance those cash flows. As a result, although the future rollover date enters in the domestic valuations, in equation (14), the rollover frequency does not enter the optimal-issuance rule, in equation (17).

In contrast to the issuance profile, the debt profile in equation (18) is decreasing in maturity. This property follows because long-term debt becomes short-term debt with time. Mathematically, the solution to PDE (8) is $f_{ss}(\tau) = \int_{\tau}^{T} \iota_{ss}(\tau')d\tau'$. Clearly, this is a function decreasing in maturity τ, since issuances are positive at all maturities.

Next, we turn to the role of impatience. The following result describes how impatience affects the steady-state amount of borrowing and its WAM.

Proposition 3 (Relative impatience). Let $\bar{\lambda} > \bar{\lambda}_o$, and assume zero-coupon bonds, $\delta = 0$. Define the relative impatience of the government as $\Delta \equiv \rho - r_{ss}^{*} > 0$. In steady state, issuances increase with relative impatience, $\partial\iota_{ss}/\partial\Delta > 0$. The elasticity with respect to relative impatience, $\epsilon_{ss,\Delta}^{\tau} = \Delta\tau/(\exp(\Delta\tau) - 1)$, is decreasing in maturity τ, and hence the WAM falls with the government's relative impatience.

As we increase impatience ρ, the government issues more steady-state debt but shortens the maturity. The increase in debt is intuitive, because a more impatient government is willing to accept lower prices at all maturities. The intuition for the reduction in the WAM is clarified by equation (19). That equation tells us what is the benefit of delaying a principal payment. That benefit is proportional to the spread $(\rho - \delta)$, which increases with impatience, discounted by time. Thus, an increase in impatience is discounted less at shorter maturities. Therefore, as we increase impatience, shorter maturities respond more, and this raises the WAM. Since coupons affect this margin, the result is ambiguous with positive bond coupons.

3. Dynamics

Next, we characterize the optimal debt-maturity management problem during transitions. During a transition, the dynamics of the optimal debt profile are dictated by two forces: expenditure smoothing and yield-curve riding.

Proposition 4 (Dynamic forces). Assume zero-coupon bonds, $\delta = 0$, and let income and interest rates revert to their steady-state values at an exponential rate α,

$$x_t = x_{ss} + (x_t - x_{ss}) \cdot \exp(-\alpha\tau) \quad \text{for } x \in \{y, r\}.$$

Then, we have the following effects.

(i) Expenditure smoothing: in response to a small negative decline in income, $\varepsilon = y_{ss} - y_0 \gtrsim 0$, when liquidity costs are arbitrarily large, $\bar{\lambda} \to \infty$, issuances increase at all maturities and the WAM decreases.

(ii) Yield-curve riding: consider a risk-neutral government, that is, $\sigma = 0$. In response to a small positive increase in interest rates, $r_0^* = r_{ss}^* + \varepsilon$, issuances decrease at all maturities and the WAM increases.

To obtain analytic tractability, proposition 4 specializes along two dimensions. First, we consider zero-coupon bonds. Second, we work with the limit as $\bar{\lambda} \to \infty$, such that the consumption path can be obtained analytically.[16] The benchmark $\bar{\lambda} \to \infty$ is a natural one because it is the case that tells us what would be the effect of shocks on valuations if the government were not to respond to the shock. Thus, this benchmark indicates the intensity with which the government wants to move its debt profile, given its desire to arbitrage. Naturally, when $\bar{\lambda} \to \infty$, the economy is in autarky (it issues no debt), an unrealistic scenario. The goal of the proposition is only to flesh out why maturity moves in a certain direction.

In the first item, we fix the short-term rate to isolate the expenditure-smoothing force. This force refers to the desire to smooth the path of expenditures. We study the effect of a temporary drop in income that translates into an identical drop in expenditures that reverts to its steady-state value. Because expenditures are expected to recover after a drop, domestic discount rates are temporarily high and then mean-revert. The elasticities of issuances with respect to the shock are positive, reflecting the desire to issue debt in order to smooth expenditures as long as the IES is positive, $\sigma > 0$. This effect on the elasticities reflects greater issuances at all maturities. In contrast to Barro (1979), with liquidity costs, expenditure smoothing affects the maturity distribution: an increasing path of expenditures produces a temporary increase in the short-term domestic discount rate r_t. Short-term debt valuations are more sensitive to the temporary increase in the discount rate. Thus, although the government increases issuances at all maturities, the increase in short-term issuances is greater in relative terms. This reduces the WAM.

In the second item, we isolate the effect of yield-curve riding. Yield-curve riding influences the optimal debt profile through bond prices, in particular, through the slope of the yield curve. To isolate this force, we set IES σ to zero so that domestic valuations $v(\tau)$ are not affected by the changes in expenditures. Just as a temporary increase in the domestic rate r_t reduces valuations, a temporary increase in the short-term rate $r_t^*(\tau)$

[16] We also employ semielasticities instead of elasticities, since the size of the shock is very small. Lemma 1 holds in both cases.

reduces market prices. From the perspective of the government, this decreases the marginal auction revenues. As a result, when short-term rates are temporarily high, the stock of debt decreases. In turn, the WAM increases because short-term prices are more sensitive to a temporary increase in the short-term rate. Hence, the optimal policy stipulates that maturity should move in the opposite direction of the slope of the yield curve, hence the term "yield-curve riding."

With a finite IES, $\sigma > 0$, changes in the short-term rate r_t^* carry effects through both yield-curve riding and expenditure smoothing. In particular, the path of rates affects the expenditure path through the financial cost of debt. Unlike smoothing, yield-curve riding is a force germane to liquidity costs. As highlighted in the frictionless benchmark above, without liquidity costs the domestic discount coincides with the short-term rate, $r_t = r_t^*$. Hence, the effects on valuations and rates are identical. All in all, the margin of adjustment is the growth rate of expenditures (and total debt), without a prediction regarding maturity.

4. Dual Problem

A final result shows that we can reinterpret our debt-management problem in terms of the typical mandates of debt-management offices. Take a given expenditure path $\{c_t\}_{t\geq 0}$, the dual of the problem in equation (10) is to minimize the net flow of financial receipts for a given expenditure path:

$$\min_{\{\iota_t(\cdot)\}} \int_0^\infty e^{-\int_0^t r(s)\, ds} \underbrace{\left(f_t(0) + \int_0^T \delta f_t(\tau)\, d\tau - \int_0^T q(\tau, t, \iota)\iota_t(\tau)\, d\tau \right)}_{\text{Net flow of financial receipts}} dt, \quad (20)$$

subject to equations (15), (8), and (7). In other words, the dual and original problems yield the same solution, provided that the expenditure paths of the dual and primal problems coincide. The proof is in appendix A9.

Debt-management offices typically have the mandate of minimizing financial expenditures, subject to a path of financing needs. The dual problem is consistent with that mandate.

IV. Quantitative Applications

In this section of the paper, we evaluate quantitative predictions of the model.

A. A Model-Based Assessment of Liquidity Costs in Spain

In a first exercise, we quantitatively evaluate our theory by contrasting implicit liquidity costs that rationalize the empirical distribution of issuances through the lens of the model with the measured liquidity costs $\bar{\lambda}(\tau)$ estimated from auction microdata in section II. To obtain model-implied liquidity costs, we extend the steady-state issuance formula in equation (18): liquidity costs are maturity dependent, as in the data. We also deviate from arbitrage-free pricing of the yield curve to be able to match the actual yield curve in the data.

We compare the steady-state issuances by maturity in the model with the average Spanish issuances over the period January 2002 to April 2018. In line with the evidence presented in section II, we modify the government's problem so that it issues only in the discrete set of maturities $\mathcal{M}$, namely, in 3, 6, 9, 12, or 18 months or in 3, 5, 10, 15, or 30 years.[17] In order to map a continuous flow of discrete issuances, $\iota_{ss}(\tau)$, with a series of monthly issuances, we integrate across time and maturity and define

$$I_{ss}(\tau) = \int_0^1 \left[\int_{\tau-\Delta t/2}^{\tau+\Delta t/2} \iota_{ss}(s) ds \right] dt,$$

as the annual issuances of bonds of maturity $\tau \in \mathcal{M}$. Integrating both sides of equation (18), we obtain:[18]

$$\bar{\lambda}(\tau) = \frac{\Delta t}{I_{ss}(\tau)} \left[1 - \frac{e^{-\rho\tau} + (\delta/\rho)(1 - e^{-\rho\tau})}{e^{-r_{ss}^*(\tau)\tau} + \left(\delta/r_{ss}^*(\tau)\right)\left(1 - e^{-r_{ss}^*(\tau)\tau}\right)} \right]. \tag{21}$$

We consider a monthly time step $\Delta t = 1/12$. This step is chosen as the minimum time period in which auctions of the different maturities are regularly issued. We calibrate the risk-free interest rates $\{r_{ss}^*(\tau)\}_{\tau\in\mathcal{M}}$ using market yields on Spanish zero-coupon bonds for the period January 2002 to April 2018. In order to build expected real rates, we subtract these nominal yields by the expected inflation derived from inflation-linked swaps (ILS) corresponding to the same maturity. The computed real interest rates are 0.9%, 1.5%, 2%, 2.3%, 2.7%, 3%, 3%, 3%, 3%, and 2.9% for the maturities of 3, 6, 9, 12, and 18 months and 3, 5, 10, 15, and 30 years, respectively. We set the (real) coupon rate δ to 2.4%, based on the average nominal coupon rate of 4.2% during the period, which we

[17] We exclude 50-year bonds because of the scarcity of issuances.

[18] Here we approximate bond prices using $\psi_{ss}(\tau) = e^{-r_{ss}^*(\tau)\tau} + \int_0^\tau \delta e^{-\int_0^s r_{ss}^*(u)\,du} ds \approx e^{-r_{ss}^*(\tau)\tau} + (\delta/r_{ss}^*(\tau))(1 - e^{-r_{ss}^*(\tau)})$. The approximation holds if the real yield curve is approximately flat, as is the case in the data except at short-term maturities. We use this approximation to price bonds at all maturities. Without this approximation, we would need to interpolate the yield curve from the points we observe in the data to all other points in the maturity domain.

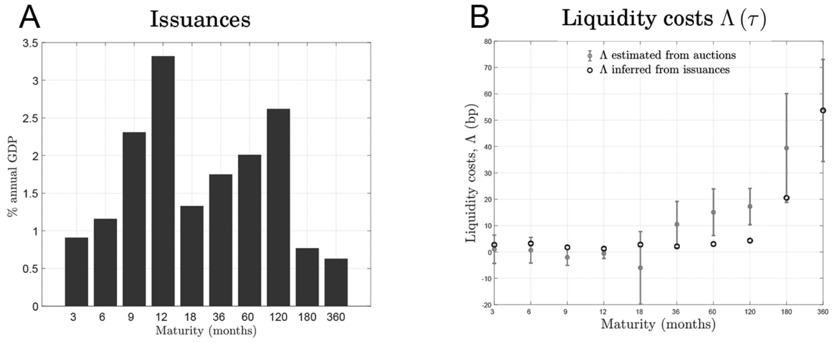

FIG. 3.—Issuance and liquidity costs by maturities. *A*, Average yearly issuances as a percentage of annual GDP. *B*, Estimates of liquidity costs. Estimates based on auction microdata include 2–standard deviation bands.

compute from the auction microdata and the average one-year-ahead inflation expectation of 1.8%, derived from ILS. We set the average income y_{ss} to 1, as a normalization.

Figure 3*A* displays the average annual issuances over GDP by maturity in the data, $I_{ss}(\tau)$. It shows how the Spanish Treasury has consistently issued significant amounts of debt at all maturities. Notwithstanding, the maturity profile has a particular shape, characterized by two humps, one with a maximum of 1 year and the other of 10 years. Issuances decrease with maturity for long-term bonds. The WAM of issuances is 4.7 years.

To map the theoretical to the estimated values in equations (7) and (3), we use a monthly aggregation:[19]

$$\frac{q_{ss}(\tau)}{\psi_{ss}(\tau)} - 1 \approx -\underbrace{\frac{1}{2}\bar{\lambda}(\tau)}_{\Lambda(\tau)} \underbrace{I_{ss}(\tau)\Delta t}_{\text{Average monthly issuances}},$$

so that the estimated liquidity costs $\Lambda(\tau)$ in figure 2 and table D1 are equivalent to $\Lambda(\tau) = (1/2)\bar{\lambda}(\tau)$, where $\bar{\lambda}(\tau)$ is the inferred value of the liquidity costs according to equation (21).

The only remaining parameter to calibrate is ρ, the government's subjective discount factor. We calibrate it to 3.3% to replicate the issuances in the longest maturity (360 months).

Figure 3*B* compares the estimated value of $\Lambda(\tau)$ based on auction microdata (lines) with the inferred values from the model using equation (21) (circles). We draw several conclusions. First, in this particular dimension our theory provides a realistic characterization of sovereign

[19] As output is normalized to 1 in the steady state, issuances are expressed as a share of average GDP.

maturity management: both liquidity measures yield similar maturity profiles. The larger discrepancies are in 5- and 10-year bonds, in which the model slightly understates liquidity costs. Second, as discussed in section II, liquidity costs grow with maturity. This increase is more linear in the auction data than in the model, which predicts a steep increase in the slope for maturities above 10 years. Third, while we cannot find evidence of liquidity costs different from zero for the maturities below 18 months (bills) in the microdata, the model predicts small positive values for these costs, within the confidence bands of the empirical estimation, which are sufficient to replicate the observed volume of issuances.

These results reassure us about the plausibility of our theory in the long run. Next we investigate its dynamic properties.

B. Dynamic Debt Management

We now describe some features of Spain's debt-management strategy along the time-series dimension and contrast these stylized facts with impulse responses computed from the model.

1. Spain's Debt-Management Strategy

Figure 4*A* depicts the time series of aggregate issuances, deficit, and the principal repayments. Spain consistently issued debt at about 10% of GDP, with a slight downward trend during the 2000s. By 2007, as Spain was hit by the Great Financial Crisis, the primary deficit began a dramatic increase, surging up to 20% of GDP. Debt repayments grew at a slower pace, as the debt increase would show up later in repayments, only after the debt matured. Whereas the deficit has fallen continually since 2009, principal amortizations continued to pile up during the sovereign debt crisis.

We run a regression on the auction data to study the determinants of total issuances and the WAM. The independent variables are (i) the primary deficit, (ii) the principal amortizations due over GDP, and (iii) the level and (iv) the slope of the yield curve, in the same quarter.[20] We summarize the regression coefficients for total issuances in figure 4*C*; values are found in table D3. The main takeaway from figure 4*C* is that issuances correlate positively with deficits and principal repayments—the correlation coefficient is approximately 0.75. Issuances are negatively correlated, although not significant, with the level and slope of the yield curve, which corresponds to the short-term rate. These figures suggest

[20] To obtain time series for the level and slope factors of the Spanish yield curve, we use estimates of a dynamic Nelson-Siegel four-factor model.

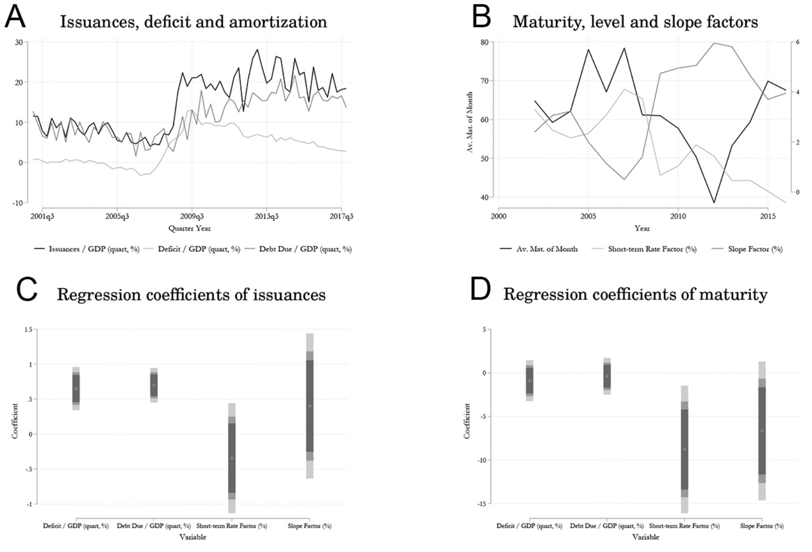

FIG. 4.—Issuance pattern. *A* depicts the quarterly debt issuances, principal amortizations, and the primary deficit as a fraction of quarterly GDP from 2001 Q1 to 2017 Q3. *B* depicts the yearly series for the average maturity in the year (weighted by issuance size) against the level and slope factors of the yield curve. *C* and *D* report the regression coefficients of quarterly issuances and quarterly maturity, respectively, against the quarterly principal amortizations, the quarterly primary deficit, and the quarterly average of the daily level and slope factors.

that overall issuances are driven by financing needs, because of both the deficit and the restructuring of debt, and, to a lesser extent, by lower interest rates.

In terms of the drivers of the WAM, figure 4*B* reports the yearly WAM, as well as the level and slope of the yield curve. We observe significant changes in the WAM during the period. These changes in the maturity composition correlate with the yield-curve factors. Figure 4*D* displays the regression coefficients. The level of short-term rates negatively correlates with the WAM, whereas the slope of the yield curve correlates positively; deficits and principal payments affect the maturity modestly. Hence, in the case of the WAM, yield-curve factors reflect strongly on it. Below, we investigate whether these correlations emerge from the model.

2. Dynamic Responses

To investigate the model's consistency with these patters, we analyze impulse responses. To this end, we follow Boppart, Krusell, and Mitman (2018), who show how the impulse response to a small shock can be computed as the perfect-foresight dynamics after an MIT shock.[21] We consider two shocks. We treat the income shock as a temporary 1% decline in income and consider a temporary 0.1% increase in the short-term rate for the interest rate shock. In both cases, we set the persistence to 0.2, which produces the same rate of mean reversion as a discrete-time AR(1) (first-order autogression) process with a quarterly persistence of 0.95. We also recalibrate the risk-free rate, r^*_{ss}, to a constant value of 3%, which is the average real rate on 10-year bonds in the Spanish sample, and the liquidity costs, $\bar{\lambda}$, to 0.34, derived from the auction-data estimation at a 10-year maturity.[22] In line with the evidence in section II, we modify the government's problem so that it issues in the discrete maturities of the set $\mathcal{M}$.[23] In this case, the government's budget constraint (eq. [9]) is

$$c_t = y_t - f_t(0) + \sum_{\tau \in \mathcal{M}} [q_t(\tau, \iota)\iota_t(\tau) - \delta f_t(\tau)]. \tag{22}$$

Naturally, the dynamics are driven by the forces we described in section III, namely, that expenditure smoothing and yield-curve riding drive the dynamic responses to these two shocks. Figure 5 depicts the impulse responses. The solid lines depict the responses to the income shock. In the case of income shocks, market prices $\psi_t(\tau)$ are constant across time. Hence, the desire to smooth expenditures is the only factor shaping the debt dynamics. Figures 5*C* and 5*D* show how the fall in revenues produces a decline in expenditures on impact, followed by a recovery. The initial fall in expenditure growth leads to an increase in the domestic discount, reverting to the steady state (fig. 5*B*). As the discount increases, valuations decrease, which acts as a temporary increase in impatience. The optimal-issuance rule (eq. [17]) dictates an increase in the issuances at all maturities (fig. 5*E*) and a decrease in the WAM (fig. 5*F*), as we should expect from proposition 4. The same pattern emerges in the simple correlations obtained from the data. This reveals that Spain's debt program is consistent with the model's qualitative prescriptions.

The responses to the interest rate shock are depicted by the dashed lines. The shock is depicted in figure 5*A*, and the rest of the panels demonstrate that the shock produces a pattern consistent with the theoretical

[21] Our algorithm builds on the upwind finite difference method of Achdou et al. (2022) to solve the bond-pricing eq. (14) and the debt distribution dynamics (eq. [8]). A description of the complete algorithm is provided in app. C.

[22] In this section, we consider a flat steady-state yield curve, as it is the one derived by the theoretical general equilibrium model in sec. III. For consistency, we also treat liquidity costs as constant. More complex arrangements do not qualitatively change the findings of this section.

[23] Namely, 3, 6, 9, 12, and 18 months and 3, 5, 10, 15, and 30 years.

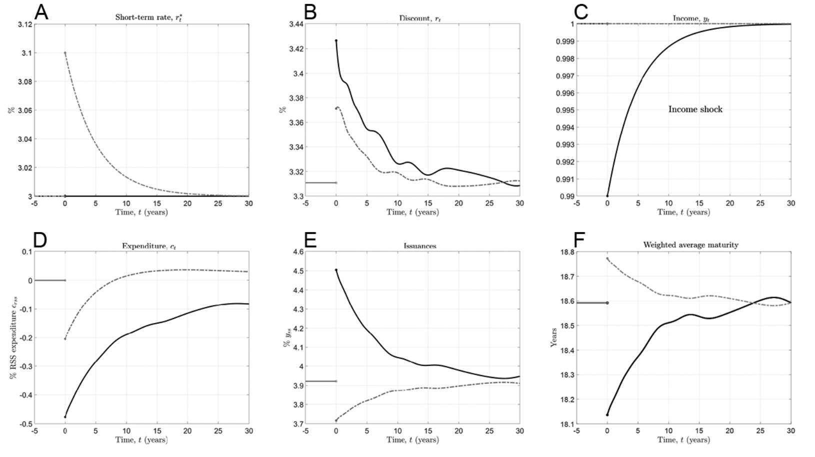

Fig. 5.—Impulse response functions to income (solid lines) and interest rate (dashed lines) shocks.

predictions of proposition 4. On impact, expenditure falls, and the domestic discount jumps, tracking the rate's path, as shown in *B* and *C*. This impact narrows the value gap across all maturities. As the gap widens, expenditures recover. The initial effect of a narrower value gap is a decrease in all issuances, as shown in *E*. This effect captures the notion that, upon an interest rate increase, the government sacrifices present expenditure to mitigate a higher debt burden. A noticeable feature is that the interest rate shock produces an increase in the WAM that occurs exactly when the yield curve slopes downward.

We have one technical comment regarding the "ripples" that can be observed in some variables, especially the discount (fig. 5*B*). They are not due to any numerical error but are a consequence of having discrete issuances: the principals mature in discrete chunks at 1, 3, 5, . . . years. These moments in time coincide with the peaks and valleys of the ripples, as they imply changes in consumption due to the budget constraint. Thus, our model predicts a cyclical pattern of responses, something that may be useful in future empirical work.

Thanks to the lessons we learn from the model, we can comment on Spain's debt management during the sovereign debt crisis. As we see in figures 4*A* and 4*B*, Spain faced a greater fiscal deficit, consistent with greater smoothing, and a more vertical yield curve. According to our model, smoothing is a force toward a shortening of the WAM, but yield-curve riding due to a (temporarily) steeper yield curve is a force in the opposite direction. Spain reduced the WAM of its issuances during the crisis, which suggests that the smoothing force dominated.

V. Additional Considerations

Up to this point, we have analyzed how, when the government internalizes the price impact of its debt issuances, expenditure smoothing and yield-curve riding emerge as forces that shape the optimal maturity distribution. Next, we investigate how other considerations, such as tax smoothing and stock effects, interact with liquidity costs.

A. *Public Finance Considerations*

In a first extension, we show that the framework above can be adapted to settings with tax distortions, as in Bhandari et al. (2017). To do so, we assume that households supply labor according to GHH (Greenwood-Hercowitz-Huffman) preferences and an inverse Frisch elasticity of ν. As is usual in the literature, the government decides on savings on behalf of households. We modify the government's problem and assume that the path of government expenditures is exogenous, $\{g_t\}$. Production is linear in hours, h_t, so the real wage is set to 1. The government sets a linear tax, η_t, on hours worked so that labor tax receipts, w_t, are given by $w_t = \eta_t h_t$. The government's budget constraint is given by

$$w_t + \int_0^T q_t(\tau)\iota_t(\tau)d\tau = g_t + \left[f_t(0) + \delta\int_0^T f_t(\tau)d\tau\right]. \tag{23}$$

The objective of the government is to maximize the household's welfare:

$$\max_{\{\iota_t(\cdot),f_t(\cdot),c_t,h_t,\eta_t\}} \int_0^\infty e^{-\rho t} U\left(c_t - \chi\frac{h_t^{1+\nu}}{1+\nu}\right)dt, \tag{24}$$

subject to the KFE (Kolmogorov forward equation; eq. [8]), the modified budget constraint (eq. [23]), the initial condition, f_0, and the optimal labor supply, $\eta_t = h_t^\nu$. The debt-management problem can be reformulated isomorphically to the version we encountered above, as we show next.

Proposition 5 (Issuances public finance). Let the domestic valuations, $v_t(\tau)$, satisfy the PDE (app. A4); let optimal issuances, $\iota_t(\tau)$, satisfy the issuance rule (eq. [13]); and suppose that the evolution of the debt distribution can be recovered from the law of motion for debt, (eq. [8]), given the initial condition f_0. Also, let the domestic rate, r_t, be given by

$$r_t = \rho - (U''/U' - W''/W')\dot{x}_t,$$

where

$$x_t \equiv y_t + \int_0^T q_t(\iota,\tau)\iota(\tau,t)\,d\tau - \left[f_t(0) + \delta\int_0^T f_t(\tau)\,d\tau\right],$$

for $y_t = -g_t$, and $W(x) \equiv \{c | c - \chi^{-1/(1+\nu)}c^{1/(1+\nu)} = x\}$ is an indirect utility. Given x_t, let

$$c_t = W(x_t),\ h_t = \left(\frac{W(x_t)}{\chi}\right)^{1/(1+\nu)},\ \text{and}\ \eta_t = 1 - \chi\left(\frac{W(x_t)}{\chi}\right)^{\nu/(1+\nu)}.$$

Then the path $\{\iota_t(\cdot), f_t(\cdot), c_t, h_t, \eta_t\}_{t\geq 0}$ induced by x_t is a solution to equation (24) if $x_t \geq -\nu(1+\nu)^{-(1+\nu)/\nu}$.

The proposition shows that the debt-management problem here can be reformulated as a problem with modified preferences, which, for the purposes of the optimal debt profile, modifies only the domestic discount rate. The discount rate itself depends on the variable x_t, which represents a private current-account deficit. This problem now admits negative income, y_t. Once we solve for the path of x_t, using the same approach as in the paper, we obtain the equilibrium consumption and labor from the indirect-utility term W. The problem does feature a constraint, a lower bound on x_t, with the interpretation that it is the point of maximal extraction of resources from the private sector, associated with the peak of the Laffer curve. We can observe that the domestic discount rate captures elements of expenditure smoothing but also smoothing of tax distortions. We do not pursue an application of optimal taxation in this paper, but we note that one can compute a solution using the algorithm discussed in this paper and obtain $\{c_t, \eta_t, h_t\}$ from proposition 5.

B. *Stock Considerations*

Section II presents evidence of liquidity costs in Spanish sovereign debt issuances. We interpreted this evidence through a wholesale-retail model of intermediation where liquidity costs can differ by maturity. We argued that this segmentation by maturity is important to rationalize the debt-issuance pattern for Spain. In practice, governments may be concerned with other forms of segmentation by maturity. In particular, governments may be concerned that the stock of outstanding debt by maturity may be relevant for the revenues raised by auctions at different maturities. There are several mechanisms that motivate this concern. As we discuss next, each of these mechanisms can have different implications for optimal debt-management practices.

1. Preferred Habitat

A reason why outstanding debt amounts at different maturities can affect auction revenues is that these outstanding amounts could affect the term structure. This is the case in models where different participants have an imperfectly elastic demand for bonds of different maturity, such as in preferred-habitat models.[24] This theory was first posited by Modigliani

[24] This does not happen in complete-market models such as those of Barro (1979), Angeletos (2002), or Buera and Nicolini (2004), where debt is priced using a common discount factor invariant to the maturity distribution of debt.

and Sutch (1967), and it has been recently reformulated in Vayanos and Vila (2021). Under preferred habitat, bond prices at maturity τ depend on the entire debt profile, $\psi_t(\tau, f_t(\cdot))$.

We can anticipate some of the forces that would shape the optimal debt profile under preferred habitat: the government will issue debt, acting like a durable-good monopolist that issues in multiple markets (see, e.g., Bulow 1982). Absent liquidity costs, the debt pattern may feature corner solutions. Just as a monopolist may sell only on markets where it obtains the highest marginal revenues, the government may prefer to issue all its bonds in the maturity markets where its debt is most inelastic. This feature would be inconsistent with the data. Another consideration under preferred habitat is time inconsistency. Time inconsistency is a key feature of sovereign default models and is the analogue of the Coase conjecture for the monopolist that sells durable goods. Bulow and Rogoff (1988) introduced the idea that if long-term debt is issued without commitment to future debt programs, investors will anticipate that future debt issuances will affect future bond prices through an increased probability of default. Thus, lack of commitment to future issuances makes long-term debt more expensive. Even without default, a similar problem emerges under the preferred-habitat theory. The reason is that the government may have incentives to announce that it will limit the supply of debt in a specific maturity in the future in order to raise current prices. Without commitment, the government will have incentives to issue debt at that maturity, precisely because its price is high when outstanding amounts are low. Naturally, lack of commitment is a problem only when prices are forward looking. If bond demand schedules depend on current quantities but are independent of future quantities, as in de Lannoy et al. (2022), preferred habitats do not induce time inconsistencies.

2. Customer Flow

Another reason why stock effects may matter is via customer flow in OTC markets. In section III.A, we assumed that the arrival rate of customers, μ, is a constant that can potentially vary by maturity but not by the outstanding amounts in the corresponding maturity. It is natural to expect that customer flows will depend on the amounts of outstanding debt at different maturities. For example, we should expect that if there is a lower supply of 30-year paper, more investors will contact dealers in search of these bonds. An extension of this paper in that direction would make μ a function of the outstanding amount at a given maturity, leading to liquidity coefficients that depend on outstanding amounts, $\lambda(f_t(\tau))$. In such an environment, the government should take into account the effect of current issuances on the customer flows to intermediaries. We anticipate that the effects on optimal issuances will lead to considerations similar to those under preferred-habitat investors.

VI. Conclusions

This paper presents a new approach to studying debt-management problems with debt instruments that resemble those that governments issue in practice. A central feature of the framework is liquidity costs that limit immediate rebalancing across maturities. The main challenge of these problems is that the state variable is a distribution. The paper presents a framework to make progress along those dimensions, highlights classic forces, and uncovers new ones that shape the optimal debt-maturity distribution.

As the first step in a new direction, the framework faces limitations. One limitation is that we abstract from anticipated risk, as the responses to shocks are computed around a deterministic steady state. A previous version of this paper also considers aggregate shocks and the option to default. That version shows that the same principles we highlight here hold—namely, the same issuance rule still applies—but that market prices and valuations must be adapted to account for both risk and default.

Finally, liquidity costs here are exogenous, but there are possible interesting feedback loops. For example, Bocola (2016) shows that the risk-bearing capacity of intermediaries is significantly hampered when default premia rise. If we interpret the capital costs of intermediaries as a limited risk-bearing capacity, it is natural to think that episodes of sovereign default risk increase liquidity costs. This feedback would lead to different predictions for maturity. We leave this for future work.

Data Availability

Code replicating the tables and figures in this paper can be found in Bigio, Nuño, and Passadore (2022), Harvard Dataverse, https://doi.org/10.7910/DVN/JWAXBR.

Appendix A

Proofs

A1. Proofs for Liquidity Cost Representation

We provide here a first-order linear approximation for the price at the auction, $q_t(\iota, \tau)$, for small issuances. The result is given by the following proposition.

Proposition 6. A first-order Taylor expansion around $\iota = 0$ yields a linear auction price:

$$q_t(\iota,\tau) \approx \psi_t(\tau) - \frac{1}{2}\frac{\eta}{\mu y_{ss}}\psi_t(\tau)\iota_t(\tau). \tag{A1}$$

Thus, the approximate liquidity cost function is $\lambda_t(\tau,\iota) \approx (1/2)\bar{\lambda}\psi_t(\tau)\iota_t(\tau)$, where the price impact is given by $\bar{\lambda} = \eta/\mu y_{ss}$.

We analyze a bond issued at time t with maturity τ. At time $t+s$, after a period of time s has passed since the auction, the time to maturity is $\tau' = \tau - s$. The valuation of the bond by investors in the secondary market is defined as

$$\psi^{(t,\tau)}(\tau', s) \equiv \psi_{t+s}(\tau - s).$$

Hence, the price equation satisfies the PDE (eq. [5]):

$$r^*_{t+s}\psi^{(t,\tau)}(\tau', s) = \delta - \frac{\partial \psi^{(t,\tau)}}{\partial \tau'} + \frac{\partial \psi^{(t,\tau)}}{\partial t},$$

with the terminal condition of $\psi^{(t,\tau)}(0,s) = 1$.

The valuation of the cash flows of the bond from the perspective of the primary dealer is $q^{(t,\tau)}(\tau', s)$. Dealers are risk neutral but have a higher cost of capital. At each moment $t+s$ dealers meet investors and sell at a price $\psi^{(t,\tau)}(\tau', s)$. The valuation of the dealers, $q^{(t,\tau)}(\tau', s)$ satisfies

$$(r^*_{t+s} + \eta)q^{(t,\tau)}(\tau', s) = \delta - \frac{\partial q^{(t,\tau)}}{\partial \tau'} + \frac{\partial q^{(t,\tau)}}{\partial t} + \gamma^{(t,\tau)}(s)\big(\psi^{(t,\tau)}(\tau', s) - q^{(t,\tau)}(\tau', s)\big). \tag{A2}$$

This expression takes this form because the dealer extracts surplus $(\psi^{(t,\tau)}(\tau', s) - q^{(t,\tau)}(\tau', s))$ when he is matched to an investor. Before a match, primary dealers earn the flow utility, but upon a match, their value jumps to $\psi^{(t,\tau)} - q^{(t,\tau)}$. This jump arrives with endogenous intensity $\gamma^{(t,\tau)}(s)$. The complication with this PDE is its terminal condition. If $\bar{s} \leq \tau$, the PDE's terminal condition is given by $q^{(t,\tau)}(\tau', \bar{s}) = \psi^{(t,\tau)}(\tau', \bar{s})$. If $\bar{s} > \tau$, the corresponding terminal condition is $q^{(t,\tau)}(0, s) = 1$, since by the expiration date, the investor is paid the principal equal to 1.

On the date of the auction, $s = 0$, $\tau' = \tau$, and the dealer pays his expected bond valuation; hence, the bond price demand faced by the government is

$$q_t(\iota,\tau) \equiv q^{(t,\tau)}(\tau, 0).$$

This is because banks have free entry into the auction.

Proof. *Step 1 (exact solutions).*—The solution to $q_t(\iota,\tau)$ falls into one of two cases.

1. If $\bar{s} \leq \tau$, then

$$q_t(\iota,\tau) = \frac{\int_0^{\bar{s}} e^{-\int_0^v \left(r^*_{t+u}+\eta\right)du}\left(\delta(\bar{s} - v) + \psi_{t+v}(\tau - v)\right)dv}{\bar{s}}. \tag{A3}$$

2. If $\bar{s} > \tau$, then

$$q_t(\iota,\tau) = \int_0^{\tau} e^{-\int_0^{s}\left(r^*_{t+u}+\eta\right)du}\left(\frac{\delta(\bar{s}-v)+\psi_{t+v}(\tau-v)}{\bar{s}}\right)dv + e^{-\int_0^{\tau}\left(r^*_{t+u}+\eta\right)du}\frac{(\bar{s}-\tau)}{\bar{s}}. \tag{A4}$$

We solve the PDE for q, depending on the corresponding terminal conditions, $q^{(t,\tau)}(\tau',\bar{s}) = \psi^{(t,\tau)}(\tau',\bar{s})$ and $q^{(t,\tau)}(0,s) = 1$.

Case 1.—Consider the first case. The general solution to the PDE for $q^{(t,\tau)}(\tau',s)$ is

$$\int_0^{\bar{s}-s} e^{-\int_0^{v}\left(r^*_{t+u}+\eta+\gamma_u\right)du}(\delta+\gamma_{s+v}\psi_{t+v}(\tau-v))dv + e^{-\int_0^{\bar{s}-s}\left(r^*_{t+u}+\eta+\gamma_u\right)du}\psi_{t+(\bar{s}-s)}(\tau'-(\bar{s}-s)). \tag{A5}$$

This can be checked by taking partial derivatives with respect to time and maturity and applying Leibniz's rule.[25] Consider the exponentials that appear in both terms of equation (A5). These can be decomposed into $e^{-\int_0^v[r^*_{t+u}+\eta]\,du}e^{-\int_0^v\gamma_u\,du}$. Then, by definition of γ we have

$$e^{-\int_0^{v}\gamma_u\,du} = e^{-\int_0^{v}[1/(\bar{s}-u)]du} = \frac{(\bar{s}-v)}{\bar{s}}. \tag{A6}$$

Thus, using equation (A6) in equation (A5), we can reexpress it as

$$q^{(t,\tau)}(\tau',s) = \int_0^{\bar{s}-s} e^{-\int_0^{v}\left(r^*_{t+u}+\eta\right)du}\frac{(\bar{s}-v)}{\bar{s}}(\delta+\gamma_{s+v}\psi_{t+v}(\tau-v))dv + e^{-\int_0^{\bar{s}-s}\left(r^*_{t+u}+\eta\right)du}\frac{s}{\bar{s}}\psi_{t+\bar{s}}(\tau'-(\bar{s}-s)).$$

When we evaluate this expression at $s=0$, $\tau'=\tau$ and replace $\gamma(v) = 1/(\bar{s}-v)$, we arrive at

$$q_t(\iota,\tau) \equiv q^{(t,\tau)}(\tau,0) = \int_0^{\bar{s}} e^{-\int_0^{v}\left(r^*_{t+u}+\eta\right)du}\left(\frac{(\bar{s}-v)}{\bar{s}}\delta+\frac{\psi_{t+v}(\tau-v)}{\bar{s}}\right)dv.$$

Case 2.—The proof in the second case runs parallel to case 1 above. The general solution to the PDE in this case is

[25] Note that we have directly replaced the value $\psi^{(t,\tau)}(\tau',s) = \psi_{t+s}(\tau-s)$.

$$q^{(\iota,\tau)}(\tau', s) = \int_0^{\tau'} e^{-\int_0^v \left(r^*_{t+u}+\eta+\gamma_u\right)du}(\delta + \gamma_{s+v}\psi_{t+v}(\tau - v))dv + e^{-\int_0^v \left(r^*_{t+u}+\eta+\gamma_u\right)du}.$$

When we evaluate this expression at $s = 0$, $\tau' = \tau$,

$$q_t(\iota, \tau) = \int_0^{\tau} e^{-\int_0^s \left(r^*_{t+u}+\eta\right)du} \frac{(\bar{s} - v)}{\bar{s}}\left(\delta + \frac{\psi_{t+v}(\tau - v)}{(\bar{s} - v)}\right)dv + e^{-\int_0^{\tau} \left(r^*_{t+u}+\eta\right)du} \frac{(\bar{s} - \tau)}{\bar{s}}.$$

Step 2 (limit behavior of $q_t(\iota, \tau)$: price with zero issuances).—Consider the limit $\iota_t(\tau) \to 0$ for any $\tau > 0$, which implies that $\bar{s} \to 0$. For both case 1 and case 2, equations (A3) and (A4),[26] it holds that

$$\lim_{\iota_t(\tau)\to 0} q_t(\iota, \tau) = \lim_{\bar{s}\to 0} \frac{\int_0^{\bar{s}} e^{-\int_0^s \left(r^*_{t+u}+\eta\right)du}(\delta(\bar{s} - s) + \psi_{t+s}(\tau - s))ds}{\bar{s}}.$$

Now, both the numerator and the denominator converge to zero as we take the limits. Hence, by L'Hôpital's rule, the limit of the price is the limit of the ratio of derivatives. The derivative of the numerator is obtained via Leibniz's rule, and thus,

$$\begin{aligned}\lim_{\iota_t(\tau)\to 0} q_t(\iota, \tau) &= \lim_{\bar{s}\to 0} \frac{\left[e^{-\int_0^s \left(r^*_{t+u}+\eta\right)du}(\delta(\bar{s} - s) + \psi_{t+s}(\tau - s))\right]\Big|_{s=\bar{s}}}{1} \\ &= \lim_{\bar{s}\to 0} e^{-\int_0^{\bar{s}} \left(r^*_{t+u}+\eta\right)du}\psi_{t+\bar{s}}(\tau - \bar{s}) \\ &= \psi_t(\tau).\end{aligned}$$

Step 3 (linear approximation of $q_t(\iota, \tau)$).—The first-order approximation of the function $q_t(\iota, \tau)$, the price at the auction, around $\iota = 0$ is given by

$$q_t(\iota, \tau) \simeq q_t(\iota, \tau)\Big|_{\iota=0} + \frac{\partial q_t(\iota, \tau)}{\partial \iota}\Big|_{\iota=0} \iota_t(\tau).$$

We computed the first term in step 2. It is given by $\psi_t(\tau)$. Thus, our objective will be to obtain $(\partial q_t(\iota, \tau)/\partial \iota)|_{\iota=0}$. Observe that by definition of $\bar{s}$, it holds that

$$\begin{aligned}\frac{\partial q_t(\iota, \tau)}{\partial \iota} &= \frac{\partial \bar{s}}{\partial \iota}\frac{\partial q_t(\iota, \tau)}{\partial \bar{s}} \\ &= \frac{1}{\mu y_{ss}}\frac{\partial q_t(\iota, \tau)}{\partial \bar{s}},\end{aligned}$$

[26] For every $\tau < \bar{s}$; i.e., in case 2, it will be analogous, since we are taking the limit when $\bar{s}$ converges to zero.

where we have applied the fact that $\bar{s} = \iota_t(\tau)/\mu y_{ss}$. For further reference, note that

$$\left.\frac{\partial q_t(\iota,\tau)}{\partial \iota}\right|_{\iota=0} = \lim_{\bar{s}\to 0}\frac{\partial q_t(\iota,\tau)}{\partial \bar{s}}\frac{1}{\mu y_{ss}}. \tag{A7}$$

Step 3.1 (derivative $\partial q_t(\iota,\tau)/\partial\bar{s}$).—Consider the price function corresponding to case 1. The derivative of the price function with respect to $\bar{s}$ is given by

$$\begin{aligned}\frac{\partial q_t(\iota,\tau)}{\partial \bar{s}} &= \frac{\partial}{\partial \bar{s}}\left(\frac{\int_0^{\bar{s}} e^{-\int_0^s \left(r^*_{t+u}+\eta\right)du}(\delta(\bar{s}-s)+\psi_{t+s}(\tau-s))ds}{\bar{s}}\right)\\
&= \frac{e^{-\int_0^{\bar{s}}(r^*_{t+u}+\eta)du}\psi(\tau-\bar{s},t+\bar{s}) + \int_0^{\bar{s}}\delta e^{-\int_0^s\left(r^*_{t+u}+\eta\right)du}ds}{\bar{s}}\\
&\quad - \frac{\int_0^{\bar{s}} e^{-\int_0^s \left(r^*_{t+u}+\eta\right)du}(\delta(\bar{s}-s)+\psi_{t+s}(\tau-s))ds}{\bar{s}^2}\\
&= \frac{e^{-\int_0^{\bar{s}}(r^*_{t+u}+\eta)du}\psi_{t+\bar{s}}(\tau-\bar{s}) + \int_0^{\bar{s}}\delta e^{-\int_0^s\left(r^*_{t+u}+\eta\right)du}ds - q_t(\iota,\tau)}{\bar{s}}.\end{aligned} \tag{A8}$$

Note that in the last line we used the definition of $q_t(\iota,\tau)$ as given for case 1.

Step 3.2 (rewriting the limit of $\partial q_t(\iota,\tau)/\partial\bar{s}$).—To obtain $(\partial q_t(\iota,\tau)/\partial\iota)|_{\iota=0}$ we compute $\lim_{\bar{s}\to 0}(\partial q_t(\iota,\tau)/\partial\bar{s})$, using equation (A8). In equation (A8), both the numerator and denominator converge to zero as $\bar{s}\to 0$.[27] Thus, we employ L'Hôpital's rule to obtain the derivative of interest. The derivative of the denominator is 1. Thus, the limit of equation (A8) is now given by

$$\lim_{\bar{s}\to 0}\frac{\partial q_t(\iota,\tau)}{\partial\bar{s}} = \lim_{\bar{s}\to 0}\frac{\partial}{\partial\bar{s}}\left[e^{-\int_0^{\bar{s}}(r^*_{t+u}+\eta)du}\psi_{t+\bar{s}}(\tau-\bar{s}) + \int_0^{\bar{s}}\delta e^{-\int_0^s\left(r^*_{t+u}+\eta\right)du}ds - q_t(\iota,\tau)\right]. \tag{A9}$$

Step 3.3.—Consider the first two terms of equation (A9). Applying Leibniz's rule,

[27] The limits of the three terms in the numerator of eq. (A8) are, respectively,

$$\lim_{\bar{s}\to 0}\int_0^{\bar{s}} e^{-\int_0^s (r(t+u)+\eta)du}ds = 0,$$

$$\lim_{\bar{s}\to 0} e^{-\int_0^{\bar{s}}(r(t+u)+\eta)du}\psi(\tau-\bar{s},t+\bar{s})) = \psi_t(\tau),$$

$$\lim_{\bar{s}\to 0} q_t(\iota,\tau) = \psi_t(\tau).$$

$$\lim_{\bar{s}\to 0}\left[\left(-\frac{\partial}{\partial \tau}\psi_{t+\bar{s}}(\tau-\bar{s})+\frac{\partial}{\partial t}\psi_{t+\bar{s}}(\tau-\bar{s})-(r^*_{t+\bar{s}}+\eta)\psi_{t+\bar{s}}(\tau-\bar{s})\right)e^{-\int_0^{\bar{s}}\left(r^*_{t+u}+\eta\right)du}+\delta e^{-\int_0^{\bar{s}}\left(r^*_{t+u}+\eta\right)du}\right].$$

The previous limit is given by

$$-\frac{\partial}{\partial \tau}\psi_t(\tau)+\frac{\partial}{\partial t}\psi_t(\tau)-(r^*{}_t+\eta)\psi_t(\tau)+\delta.$$

Using the valuation of the international investors, we can rewrite the previous equation as

$$\begin{aligned}-\frac{\partial}{\partial \tau}\psi_t(\tau)+\frac{\partial}{\partial t}\psi_t(\tau)-(r^*{}_t+\eta)\psi_t(\tau)+\delta &= r^*{}_t\psi_t(\tau)-(r^*{}_t+\eta)\psi_t(\tau)\\ &= -\eta\psi_t(\tau).\end{aligned}\tag{A10}$$

Thus, the first two terms of the limit of $\partial q_t(\iota,\tau)/\partial\bar{s}$ are equal to $-\eta\psi_t(\tau)$. The last term of equation (A9) is given by

$$\begin{aligned}-\lim_{\bar{s}\to 0}\frac{\partial q_t(\iota,\tau)}{\partial \bar{s}} &= -\lim_{\bar{s}\to 0}\frac{\partial q_t(\iota,\tau)}{\partial \iota}\frac{\partial \iota}{\partial \bar{s}}\\ &= -\left.\frac{\partial q_t(\iota,\tau)}{\partial \iota}\right|_{\iota=0}\mu y_{ss},\end{aligned}\tag{A11}$$

where we used equation (A7). Thus, from equations (A10) and (A11), the derivative (eq. [A8]) is given by

$$\lim_{\bar{s}\to 0}\frac{\partial q_t(\iota,\tau)}{\partial \bar{s}} = -\left.\frac{\partial q_t(\iota,\tau)}{\partial \iota}\right|_{\iota=0}\mu y_{ss}-\eta\psi_t(\tau).\tag{A12}$$

Plugging equation (A12) into equation (A7), we obtain that

$$\left.\frac{\partial q_t(\iota,\tau)}{\partial \iota}\right|_{\iota=0} = \left(-\mu y_{ss}\left.\frac{\partial q_t(\iota,\tau)}{\partial \iota}\right|_{\iota=0}-\eta\psi_t(\tau)\right)\frac{1}{\mu y_{ss}}.$$

Rearranging terms, we conclude that

$$\left.\frac{\partial q_t(\iota,\tau)}{\partial \iota}\right|_{\iota=0} = -\frac{\eta\psi_t(\tau)}{2\mu y_{ss}}.\tag{A13}$$

Step 4 (Taylor expansion).—A first-order Taylor expansion around zero emissions yields

$$\begin{aligned}q_t(\iota,\tau) &\simeq q_t(\iota,\tau)\Big|_{\iota=0}+\left.\frac{\partial q_t(\iota,\tau)}{\partial \iota}\right|_{\iota=0}\iota_t(\tau)\\ &= \psi_t(\tau)-\frac{\eta\psi_t(\tau)}{2\mu y_{ss}}\iota_t(\tau),\end{aligned}$$

where we used equation (A13). We can define price impact as $\bar{\lambda}=\eta/\mu y_{ss}$. This concludes the proof. QED

A2. *Proof of Proposition 1*

First, we construct a Lagrangian on the space of functions g that are Lebesgue integrable, $\| e^{-\rho t/2} g_t(\tau) \|^2 < \infty$. The Lagrangian, after replacement of c_t from the budget constraint, is

$$\mathcal{L}[\iota, f] = \int_0^\infty e^{-\rho t} U\left(y_t - f_t(0) + \int_0^T [q(\tau, t, \iota)\iota_t(\tau) - \delta f_t(\tau)]\, d\tau \right) dt + \int_0^\infty \int_0^T e^{-\rho t} j_t(\tau) \left(-\frac{\partial f}{\partial t} + \iota_t(\tau) + \frac{\partial f}{\partial \tau} \right) d\tau\, dt,$$

where $j_t(\tau)$ is the Lagrange multiplier associated to the law of motion of debt.

We consider a perturbation $h_t(\tau)$, $e^{-\rho t} h \in L^2([0, T] \times [0, \infty))$, around the optimal solution. Since the initial distribution f_0 is given, any feasible perturbation must satisfy $h_0(\tau) = 0$. In addition, we know that $f_t(T) = 0$. Thus, any admissible variation must also feature $h_t(T) = 0$. At an optimal solution f, the Lagrangian must satisfy $\mathcal{L}[\iota, f] \geq \mathcal{L}[\iota, f + \alpha h]$ for any perturbation $h_t(\tau)$.

Taking the derivative with respect to α—that is, computing the Gâteaux derivative, for any suitable $h_t(\tau)$ we obtain

$$\begin{aligned} \frac{d}{d\alpha} \mathcal{L}[\iota, f + \alpha h] \bigg|_{\alpha=0} &= \int_0^\infty e^{-\rho t} U'(c_t) \left[-h_t(0) - \int_0^T \delta h_t(\tau)\, d\tau \right] dt \\ &\quad - \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j_t(\tau) d\tau\, dt \\ &\quad + \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial \tau} j_t(\tau) d\tau\, dt. \end{aligned}$$

We employ integration by parts to show that

$$\begin{aligned} \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j_t(\tau) d\tau\, dt &= \int_0^T \int_0^\infty e^{-\rho t} \frac{\partial h}{\partial t} j_t(\tau) dt\, d\tau \\ &= \int_0^T \left(\lim_{s \to \infty} e^{-\rho s} h_s(\tau) j_s(\tau)] - h_0(\tau) j_0(\tau) \right) d\tau \\ &\quad - \int_0^T \int_0^\infty e^{-\rho t} \left(\frac{\partial j_t(\tau)}{\partial t} - \rho j_t(\tau) \right) h_t(\tau) dt\, d\tau, \end{aligned}$$

and

$$\int_0^\infty e^{-\rho t} \int_0^T \frac{\partial h}{\partial \tau} j_t(\tau) d\tau\, dt = \int_0^\infty e^{-\rho t} \left[h_t(T) j_t(T) - h_t(0) j_t(0) - \int_0^T h_t(\tau) \frac{\partial j}{\partial \tau}\, d\tau \right] dt.$$

Replacing these calculations in the Lagrangian and equating it to zero yields

$$
\begin{aligned}
0 = & \int_0^\infty e^{-\rho t} U'(c_t)\left[-h_t(0) - \int_0^T \delta h_t(\tau)\, d\tau\right] dt \\
& + \int_0^\infty \int_0^T e^{-\rho t}\left(-\rho j - \frac{\partial j}{\partial \tau} + \frac{\partial j}{\partial t}\right) h_t(\tau) d\tau\, dt \\
& + \int_0^\infty e^{-\rho t}(h_t(T) j_t(T) - h_t(0) j_t(0)) dt \\
& - \int_0^\infty \lim_{s\to\infty} e^{-\rho s} h_s(\tau) j_s(\tau)\, d\tau + h_0(\tau) j_0(\tau).
\end{aligned}
$$

We rearrange terms to obtain

$$
\begin{aligned}
0 = & -\int_0^\infty e^{-\rho t}[U'(c_t) - j_t(0)] h_t(0) dt \\
& + \int_0^\infty \int_0^T e^{-\rho t}\left(-\rho j - U'(c)\delta - \frac{\partial j}{\partial \tau} + \frac{\partial j}{\partial t}\right) h_t(\tau) d\tau\, dt \\
& - \int_0^\infty e^{-\rho t}(h_t(T) j_t(T)) dt \\
& - \int_0^\infty \lim_{s\to\infty} e^{-\rho s} h_s(\tau) j_s(\tau) d\tau + h_0(\tau) j_0(\tau).
\end{aligned}
\tag{A14}
$$

Since $h_t(T) = h_0(\tau) = 0$ is a condition for any admissible variation, then, both the third line in equation (A14) and the second term in the fourth line are equal to zero. Furthermore, because equation (A14) has to hold for any feasible variation $h_t(\tau)$, all the terms that multiply $h_t(\tau)$ should equal zero. The latter yields a system of necessary conditions for the Lagrange multipliers:

$$
\begin{aligned}
\rho j_t(\tau) &= -\delta U'(c_t) + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau}, \quad \text{if } \tau \in (0, T], \\
j_t(0) &= -U'(c_t), \quad \text{if } \tau = 0, \\
\lim_{t\to\infty} e^{-\rho t} j_t(\tau) &= 0, \quad \text{if } \tau \in (0, T].
\end{aligned}
\tag{A15}
$$

Next, we perturb the control. We proceed in a similar fashion:

$$
\begin{aligned}
\frac{d}{d\alpha}\mathcal{L}[\iota + \alpha h, f]|_{\alpha=0} = & \int_0^\infty e^{-\rho t} U'(c_t)\left[\int_0^T \left(\frac{\partial q}{\partial \iota}\iota_t(\tau) + q_t(\tau, \iota)\right) h_t(\tau) d\tau\right] dt \\
& + \int_0^\infty \int_0^T e^{-\rho t} h_t(\tau) j_t(\tau) d\tau\, dt.
\end{aligned}
$$

Collecting terms and setting the Lagrangian to zero, we obtain

$$
\int_0^\infty \int_0^T e^{-\rho t}\left[j_t(\tau) + U'(c_t)\left(\frac{\partial q}{\partial \iota}\iota_t(\tau) + q_t(\tau, \iota)\right)\right] h_t(\tau) d\tau\, dt = 0.
$$

Thus, setting the term in parentheses to zero amounts to setting

$$U'(c_t)\left(\frac{\partial q}{\partial \iota}\iota_t(\tau) + q_t(\tau, \iota)\right) = -j_t(\tau). \quad \text{(A16)}$$

Next, we define the Lagrange multiplier in terms of goods:

$$v_t(\tau) = -j_t(\tau)/U'(c_t). \quad \text{(A17)}$$

Taking the derivative of $v_t(\tau)$ with respect to t and τ, we can express the necessary conditions, equation (A15), in terms of v. In particular, we transform the PDE in equation (A15) into the summary equations in the proposition. That is,

$$\left(\rho - \frac{U''(c_t)c_t}{U'(c_t)}\frac{\dot{c}_t}{c_t}\right)v_t(\tau) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \quad \text{if } \tau \in (0, T],$$
$$v_t(0) = 1, \quad \text{if } \tau = 0,$$
$$\lim_{t\to\infty} e^{-\rho t} v_t(\tau) = 0, \quad \text{if } \tau \in (0, T],$$

and the first-order condition, equation (A16), is now given by

$$\frac{\partial q}{\partial \iota}\iota_t(\tau) + q_t(\tau, \iota) = v_t(\tau),$$

as we intended to show. QED

A3. Proof of Lemma 1

We establish the sign relationship between ϵ^{μ}_{θ} and $\partial\epsilon^{\tau}_{t,\theta}/\partial\tau$. First, observe that if $\iota_t(s) > 0$, for all $\tau \in [0, T]$,

$$\mu_t = \int_0^T \tau \frac{\iota_t(\tau)}{\int_0^T \iota_t(z)dz} d\tau$$

is an expectation: $\mu_t = \mathbb{E}_{g_t}[\tau]$ under

$$g_t(\tau) \equiv \frac{\iota_t(\tau)}{\int_0^T \iota_t(z)dz},$$

a density function in $s \in [0, T]$. Let

$$G_t(\tau) \equiv \int_0^{\tau} g_t(s)ds$$

be the cumulative distribution associated with g. We index $\{G, g\}$ by θ, a parameter of interest that affects issuances in a comparative statics. Naturally, these distributions move continuously with θ. The following argument invokes first-order stochastic dominance. Consider two arbitrary values of the parameter of interest, θ and θ', such that $\theta' > \theta$. By definition, μ_t is increasing in θ if and only if

$$\mu_t(\theta) = \mathbb{E}_{g_t(\theta)}[\tau] \leq \mathbb{E}_{g_t(\theta')}[\tau] = \mu_t(\theta').$$

Observe that since τ is increasing, then, by definition of first-order stochastic dominance, the condition above is identical to

$$G(\tau;\theta) \geq G(\tau;\theta')$$

for all $\tau \in [0, T]$. Because ι is a continuous and bounded function of θ, this condition is equivalent to the local condition:

$$\frac{\partial}{\partial\theta}[G(\tau;\theta)] \leq 0, \quad \forall\, \tau \in [0, T].$$

Next, we translate the conditions on G into a condition related to the elasticity of issuances. Observe that

$$\frac{\partial}{\partial\theta}[G(\tau;\theta)] = G(\tau;\theta)\left[\frac{\int_0^\tau (\partial/\partial\theta)[\iota_t(s)]ds}{\int_0^\tau \iota_t(s)ds} - \frac{\int_0^T (\partial/\partial\theta)[\iota_t(s)]ds}{\int_0^T \iota_t(s)ds}\right].$$

Thus, since the term outside the brackets is positive, the sign of $(\partial/\partial\theta)[G(\tau;\theta)]$ depends on the sign of the term inside the brackets. Thus, $(\partial/\partial\theta)[G(\tau;\theta)] \leq 0$ is equivalent to

$$\frac{\int_0^\tau (\partial/\partial\theta)[\iota_t(s)]ds}{\int_0^\tau \iota_t(s)ds} \leq \frac{\int_0^T (\partial/\partial\theta)[\iota_t(s)]ds}{\int_0^T \iota_t(s)ds}, \quad \forall\, \tau \in [0, T].$$

To aid the calculations, we define the auxiliary function

$$H_t(\tau) \equiv \frac{\int_0^\tau (\partial/\partial\theta)[\iota_t(s)]ds}{\int_0^\tau \iota_t(s)ds}$$

and express the condition as

$$H_t(\tau) \leq H_t(T) \quad \forall\, \tau \in [0, T].$$

This is a necessary and sufficient condition for monotone comparative statics about the WAM.

Next, we obtain a stronger sufficient condition. Taking the derivative

$$
\begin{aligned}
\frac{\partial}{\partial \tau} H_t(\tau) &= H_t(\tau)\left[\frac{\iota_{t,\theta}(\tau)}{\int_0^\tau \iota_{t,\theta}(s)ds} - \frac{\iota_t(\tau)}{\int_0^\tau \iota_t(s)ds}\right] \\
&= \frac{\iota_t(\tau)}{\theta}\frac{H_t(\tau)}{\int_0^\tau \iota_{t,\theta}(s)ds}\left[\frac{\iota_{t,\theta}(\tau)}{\iota_t(\tau)}\theta - \frac{\int_0^\tau \iota_{t,\theta}(s)ds}{\int_0^\tau \iota_t(s)ds}\theta\right] \\
&= \frac{\iota_t(\tau)}{\theta}\frac{\left(\int_0^\tau \iota_{t,\theta}(s)ds/\int_0^\tau \iota_t(s)ds\right)}{\int_0^\tau \iota_{t,\theta}(s)ds}\left[\frac{\iota_{t,\theta}(\tau)}{\iota_t(\tau)}\theta - \frac{\int_0^\tau \iota_{t,\theta}(s)ds}{\int_0^\tau \iota_t(s)ds}\theta\right] \\
&= \frac{\iota_t(\tau)}{\int_0^\tau \theta\iota_t(s)ds}\left[\frac{\iota_{t,\theta}(\tau)}{\iota_t(\tau)}\theta - \frac{\int_0^\tau \iota_{t,\theta}(s)ds}{\int_0^\tau \iota_t(s)ds}\theta\right] \\
&= \frac{\iota_t(\tau)}{\int_0^\tau \theta\iota_t(s)ds}\left[\frac{\iota_{t,\theta}(\tau)}{\iota_t(\tau)}\theta - \int_0^\tau \frac{\iota_{t,\theta}(s)\theta}{\iota_t(s)}\frac{\iota_t(s)}{\int_0^\tau \iota_t(z)dz}\,ds\right] \\
&= \frac{\iota_t(\tau)}{\int_0^\tau \theta\iota_t(s)\,ds}\left[\epsilon_{t,\theta}^\tau - \mathbb{E}_{g_t(\theta')}\left[\epsilon_{t,\theta}^s | s < \tau\right]\right],
\end{aligned}
$$

where $\iota_{t,\theta}(\tau) \equiv \partial\iota_t(\tau)/\partial\theta$. The term outside the brackets is positive by assumption. Thus,

$$
\operatorname{sign}\left(\frac{\partial}{\partial\tau}H_t(\tau)\right) = \operatorname{sign}\left(\epsilon_{t,\theta}^\tau - \mathbb{E}_{g_t(\theta')}[\epsilon_{t,\theta}^s | s < \tau]\right).
$$

If $\epsilon_{t,\theta}^\tau$ is increasing in τ, then $\epsilon_{t,\theta}^\tau - \mathbb{E}_{g_t(\theta')}[\epsilon_{t,\theta}^s | s < \tau] > 0$, and hence $(\partial/\partial\tau)H_t(\tau) > 0$ and μ_t increases with θ. The reverse result applies if $\epsilon_{t,\theta}^\tau$ is decreasing. QED

A4. Frictionless Benchmark

Assume that $\lambda_t(\tau, \iota) = 0$. If a solution exists, then consumption satisfies equation (15) with $r_t^* = r_t$ and the initial condition

$$
B_0 = \int_0^\infty \exp\left(-\int_0^s r_u^*\,du\right)(c_s - y_s)ds.
$$

Given the optimal path of consumption, any solution $\iota_t(\tau)$ consistent with equation (8) and

$$
\dot{B}_t = r^*{}_t B_t + c_t - y_t, \quad \text{for } t > 0, \tag{A18}
$$

where

$$B_t = \int_0^T \psi_t(\tau) f_t(\tau) d\tau, \tag{A19}$$

is an optimal solution.

Step 1.—The first part of the proof is just a direct consequence of the first-order condition $v_t(\tau) = \psi_t(\tau)$ for bond issuance. Bond prices are given by equation (5), while the government valuations are given by equation (14). Since both equations must be equal in a bounded solution, we conclude that

$$r_t^* = r_t = \rho - \frac{U''(c_t)}{U'(c_t)} \frac{dc}{dt}$$

must describe the dynamics of consumption.

Step 2.—The second part of the proof derives the law of motion of B_t. First, we take the derivative with respect to time at both sides of definition (A19). Recall that, from the law of motion of debt, equation (8), it holds that

$$\iota_t(\tau) = -\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \tau}.$$

To express the budget constraint in terms of f, we substitute $\iota_t(\tau)$ into the budget constraint:

$$c_t = y_t - f_t(0) + \int_0^T \left[\psi_t(\tau) \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial \tau} \right) - \delta f_t(\tau) \right] d\tau. \tag{A20}$$

We would like to rewrite equation (A20). Therefore, first, we apply integration by parts to the following expression:

$$\int_0^T \psi_t(\tau) \frac{\partial f}{\partial \tau} d\tau = \psi_t(T) f_t(T) - \psi_t(0) f_t(0) - \int_0^T \frac{\partial \psi}{\partial \tau} f_t(\tau) d\tau.$$

As long as the solution is smooth, it holds that $f_t(T) = 0$. Further, recall that, by construction, $\psi_t(0) = 1$. Hence,

$$\int_0^T \psi_t(\tau) \frac{\partial f}{\partial \tau} d\tau = -f_t(0) - \int_0^T \frac{\partial \psi}{\partial \tau} f_t(\tau) d\tau. \tag{A21}$$

Second, from the pricing equation of international investors, we know that

$$\frac{\partial \psi}{\partial \tau} = -r^*{}_t \psi_t(\tau) + \delta + \frac{\partial \psi}{\partial t}.$$

Then, we obtain

$$\int_0^T \psi_t(\tau) \frac{\partial f}{\partial \tau} d\tau = -f_t(0) - \int_0^T [\delta + \psi_t(\tau) - r_t \psi_t(\tau)] f_t(\tau) d\tau. \tag{A22}$$

We substitute equations (A21) and (A22) into equation (A20), and thus,

$$\begin{aligned} c_t &= y_t - f_t(0) + \int_0^T \left[\psi_t(\tau)\frac{\partial f}{\partial t} - \delta f_t(\tau)\right]d\tau \ldots \\ &- \left\{-f_t(0) - \int_0^T [\delta + \psi_t(\tau) - r_t\psi_t(\tau)]f_t(\tau)d\tau\right\} \\ &= y_t + \int_0^T [\dot{\psi}_t(\tau)f_t(\tau) + \psi_t(\tau)\dot{f}_t(\tau)]d\tau - \int_0^T r^*{}_t\psi_t(\tau)f_t(\tau)d\tau. \end{aligned}$$

Rearranging terms and employing the definitions above, we obtain

$$\dot{B}_t = c_t - y_t + r^*{}_t B_t,$$

as desired.

A5. Asymptotic Behavior

Here we formally prove the limit conditions of proposition 1. In particular, we provide a complete asymptotic characterization. The following proposition provides a summary.

PROPOSITION 7. Assume that $\rho > r^*_{ss}$; then there exists a steady state if and only if $\bar{\lambda} > \bar{\lambda}_o$ for some $\bar{\lambda}_o$. If instead, $\bar{\lambda} \le \bar{\lambda}_o$, there is no steady state, but consumption converges asymptotically to zero. In particular, the asymptotic behavior is as follows. *Case 1 (High liquidity costs).*—For liquidity costs above the threshold value $\bar{\lambda} > \bar{\lambda}_o$, variables converge to a steady state characterized by the following system:

$$\frac{\dot{c}_{ss}}{c_{ss}} = 0,$$

$$r_{ss} = 0, \tag{A23}$$

$$\iota_{ss}(\tau) = \frac{\psi_{ss}(\tau) - v_{ss}(\tau)}{\bar{\lambda}\psi_{ss}(\tau)}, \tag{A24}$$

$$v_{ss}(\tau) = \frac{\delta}{\rho}(1 - e^{-\rho\tau}) + e^{-\rho\tau}, \tag{A25}$$

$$f_{ss}(\tau) = \int_\tau^T \iota_{ss}(s)ds, \tag{A26}$$

$$c_{ss} = y_{ss} - f_{ss}(0) + \int_0^T \left[\psi_{ss}(\tau)\iota_{ss}(\tau) - \frac{\bar{\lambda}\psi_{ss}(\tau)}{2}\iota_{ss}(\tau)^2 - \delta f_{ss}(\tau)\right]. \tag{A27}$$

Case 2 (Low liquidity costs).—For liquidity costs below the threshold value $0 < \bar{\lambda} \le \bar{\lambda}_o$, variables converge asymptotically to

$$\begin{aligned} \lim_{s\to\infty}\frac{c_s}{c_t} &= e^{-(\rho - r_\infty(\bar{\lambda}))(s-t)/\sigma}, \\ v_\infty(\tau, r_\infty(\bar{\lambda})) &= \frac{\delta}{r_\infty(\bar{\lambda})}(1 - e^{-r_\infty(\bar{\lambda})\tau}) + e^{-r_\infty(\bar{\lambda})\tau}, \\ \iota_\infty(\tau, r_\infty(\bar{\lambda})) &= \frac{\psi_{ss}(\tau) - v_\infty(\tau, r_\infty(\bar{\lambda}))}{\bar{\lambda}\psi_{ss}(\tau)}, \\ f_\infty(\tau, r_\infty(\bar{\lambda})) &= \int_\tau^T \iota_\infty(s, r_\infty(\bar{\lambda}))ds, \end{aligned}$$

where $r_\infty(\bar{\lambda})$ satisfies $r_{ss}^* \leq r_\infty(\bar{\lambda}) < \rho$ and solves

$$\begin{aligned} c_\infty &= 0 \\ &= y_{ss} - f_\infty(0, r_\infty(\bar{\lambda})) + \int_0^T \left[\iota_\infty(\tau, r_\infty(\bar{\lambda}))\psi(\tau) - \frac{\bar{\lambda}\psi_{sst}(\tau)}{2}\iota_\infty(\tau, r_\infty(\bar{\lambda}))^2 - \delta f_\infty(\tau, r_\infty(\bar{\lambda}))\right] d\tau. \end{aligned}$$

Threshold.—The threshold $\bar{\lambda}_o$ solves $|c_{ss}|_{\bar{\lambda}=\bar{\lambda}_o} = 0$ in equation (A27) and $\lim_{\bar{\lambda}\to\bar{\lambda}_o} r_\infty(\bar{\lambda}) = \rho$.

Proof. *Step 1.*—First, observe that as $\bar{\lambda}\to\infty$, the optimal-issuance policy (eq. [17]) approaches $\iota_t(\tau) = 0$. Thus, for that limit, $c_{ss} = y > 0$ and $f_{ss}(\tau) = 0$.

Step 2.—Next, consider the system in case 1 of proposition 7 as a guess of a solution. Note that equations (A24)–(A27) meet the necessary conditions of proposition 1 as long as $r_t = \rho$. This is because $\iota_{ss}(\tau)$ meets the first-order condition with respect to the control; $v_{ss}(\tau)$ solves the PDE for valuations; given $\iota_{ss}(\tau)$ and $v_{ss}(\tau)$, the stock of debt solves the KFE and thus is given by $\int_\tau^T \iota_{ss}(s)\,ds$; and consumption is pinned down by the budget constraint. In addition, by construction, consumption determined in equation (A27) does not depend on time; that is, $\dot{c}_t = 0$, and this implies that

$$r_{ss} \equiv r_t = \rho.$$

Thus, the only thing we need to check is that there exists some $\bar{\lambda}$ finite such that consumption is positive.

Step 3.—The system in equations (A24)–(A27) is continuous in $\bar{\lambda}$. Therefore, because $c_{ss} = y > 0$ for $\bar{\lambda}\to\infty$, there exists a value of $\bar{\lambda}$ such that the implied consumption by equations (A24)–(A27) is positive.

Step 4.—We now prove that there is an interval where this solution holds. In particular, we show that c_{ss} decreases as $\bar{\lambda}$ increases. Observe that steady-state internal valuations $v_{ss}(\tau)$ in equation (A25) and bond prices $\psi(\tau)$ are independent of $\bar{\lambda}$. Steady-state debt issuances $\iota_{ss}(\tau)$ in equation (A24) are a monotonously decreasing function of $\bar{\lambda}$, because

$$\frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} = -\frac{1}{\bar{\lambda}}\iota_{ss}(\tau) < 0,$$

and therefore the total amount of debt at each maturity $f_{ss}(\tau)$ in equation (A26) is also decreasing with $\bar{\lambda}$, because

$$\frac{\partial f_{ss}(\tau)}{\partial \bar{\lambda}} = -\frac{1}{\bar{\lambda}} f_{ss}(\tau) < 0.$$

If we take derivatives with respect to $\bar{\lambda}$ in the budget constraint (eq. [A27]), we obtain

$$\begin{aligned} \frac{\partial c_{ss}}{\partial \bar{\lambda}} &= -\frac{\partial f_{ss}(0)}{\partial \bar{\lambda}} + \int_0^T \left[\psi_{ss}(\tau)\frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} - \frac{\psi_{ss}(\tau)}{2}\iota_{ss}(\tau)^2 - \bar{\lambda}\psi_{ss}(\tau)\iota_{ss}(\tau)\frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} - \delta\frac{\partial f_{ss}(\tau)}{\partial \bar{\lambda}}\right] d\tau \\ &= \frac{1}{\bar{\lambda}} f_{ss}(0) - \frac{1}{\bar{\lambda}}\int_0^T \left[\psi_{ss}(\tau)\iota_{ss}(\tau) + \bar{\lambda}\frac{\psi_{ss}(\tau)}{2}\iota_{ss}(\tau)^2 - \bar{\lambda}\psi_{ss}(\tau)\iota_{ss}(\tau)^2 - \delta f_{ss}(\tau)\right] d\tau \\ &= -\frac{1}{\bar{\lambda}} c_{ss} < 0. \end{aligned}$$

Observe that $\iota_{ss}(\tau)$ can be made arbitrarily small by increasing $\bar{\lambda}$. Thus, there exists a value of $\bar{\lambda} \geq 0$ such that $c_{ss} = 0$ in the system above. We denote this value by $\bar{\lambda}_o$.

Step 5.—For $\bar{\lambda} \leq \bar{\lambda}_o$, if a steady state existed, it would imply $c_{ss} < 0$, outside of the range of admissible values. Therefore, there is no steady state in this case. Assume that the economy grows asymptotically at rate $g_\infty(\bar{\lambda}) \equiv \lim_{t\to\infty}(1/c_t)(dc/dt)$. If $g_\infty(\bar{\lambda}) > 0$, then consumption would grow to infinity, which violates the budget constraint. Thus, if there exists an asymptotic the growth rate, it is negative: $g_\infty(\bar{\lambda}) < 0$. If we define $r_\infty(\bar{\lambda})$ as

$$r_\infty(\bar{\lambda}) \equiv (\rho + \sigma g(\bar{\lambda})) < \rho,$$

the growth rate of the economy can be expressed as

$$g_\infty(\bar{\lambda}) = -\frac{(\rho - r_\infty(\bar{\lambda}))}{\sigma}.$$

When this is the case, the asymptotic valuation is

$$v_\infty(\tau, r_\infty(\bar{\lambda})) = \frac{\delta\left(1 - e^{-r_\infty(\bar{\lambda})\tau}\right)}{r_\infty(\bar{\lambda})} + e^{-r_\infty(\bar{\lambda})\tau}.$$

To obtain the discount factor bounds, observe that if $v_\infty(\tau, r_\infty(\bar{\lambda})) \leq \psi_{ss}(\tau)$, the optimal issuance is nonnegative. Otherwise, issuances would be negative at all maturities and the country would be an asymptotic net asset holder. This cannot be an optimal solution, as this implies that consumption can be increased just by reducing the amount of foreign assets. Therefore, $r_\infty(\bar{\lambda}) \geq r^*$. Finally, by definition, $r_\infty(\bar{\lambda}) < \rho$. QED

A6. Limiting Distribution: $\bar{\lambda} \to 0$

Proposition 8 (Limiting distribution). In the limit as liquidity costs vanish, $\bar{\lambda} \to 0$, the asymptotic optimal issuance is given by

$$\iota_\infty^{\bar{\lambda}\to 0}(\tau) = \lim_{\bar{\lambda}\to 0} \iota_\infty(\tau) = \frac{1 + \left[-1 + (r^*/\delta - 1) r_{ss}^* \tau\right] e^{-r_{ss}^* \tau}}{1 + \left[-1 + (r^*/\delta - 1) r_{ss}^* T\right] e^{-r_{ss}^* T}} \frac{\psi_{ss}(T)}{\psi_{ss}(\tau)} \Xi, \tag{A28}$$

where constant $\Xi > 0$ is such that $y_{ss} - f_\infty^{\bar{\lambda}\to 0}(0) + \int_0^T [\iota_\infty^{\bar{\lambda}\to 0}(\tau)\psi_{ss}(\tau) - \delta f_\infty^{\bar{\lambda}\to 0}(\tau)] d\tau = 0$ and $f_\infty^{\bar{\lambda}\to 0}(\tau) = \int_\tau^T \iota_\infty^{\bar{\lambda}\to 0}(s) ds$.

Proof. Consider the following limit:

$$\begin{aligned}
\iota_\infty^{\bar{\lambda}\to 0}(\tau) &\equiv \lim_{\bar{\lambda}\to 0} \iota_\infty(\tau, r_\infty(\bar{\lambda})) \\
&= \lim_{\bar{\lambda}\to 0} \frac{\psi_{ss}(\tau) - v_\infty(\tau, r_\infty(\bar{\lambda}))}{\bar{\lambda}\psi_{ss}(\tau)} \\
&= \lim_{\bar{\lambda}\to 0} \frac{1}{\bar{\lambda}\psi_{ss}(\tau)} \left[\frac{\delta(1 - e^{-r_{ss}^*\tau})}{r_{ss}^*} - \frac{\delta(1 - e^{-r_\infty(\bar{\lambda})\tau})}{r_\infty(\bar{\lambda})} + e^{-r_{ss}^*\tau} - e^{-r_\infty(\bar{\lambda})\tau}\right].
\end{aligned}$$

This is a limit of the form 0/0, as $\lim_{\bar{\lambda}\to 0} r_\infty(\bar{\lambda}) = r^*$.[28] We do not have an expression for $r_\infty(\bar{\lambda})$, so we cannot apply L'Hôpital's rule directly. Instead, we compute

$$\lim_{\bar{\lambda}\to 0} \frac{\iota_\infty(\tau, r_\infty(\bar{\lambda}))}{\iota_\infty(T, r_\infty(\bar{\lambda}))}$$
$$= \lim_{r_\infty(\bar{\lambda})\to r^*} \frac{\left[\delta\left(1 - e^{-r^*\tau}\right)/r^*\right] - \left[\delta\left(1 - e^{-r_\infty(\bar{\lambda})\tau}\right)/r\right] + e^{-r^*\tau} - e^{-r_\infty(\bar{\lambda})\tau}}{\left[\delta\left(1 - e^{-r^*T}\right)/r^*\right] - \left[\delta\left(1 - e^{-r_\infty(\bar{\lambda})T}\right)/r\right] + e^{-r^*T} - e^{-r_\infty(\bar{\lambda})T}} \frac{\psi_{ss}(T)}{\psi_{ss}(\tau)},$$

which also has a limit of the form 0/0. Now we can apply L'Hôpital's rule. We obtain

$$\lim_{\bar{\lambda}\to 0} \frac{\iota_\infty(\tau, r_\infty(\bar{\lambda}))}{\iota_\infty(T, r_\infty(\bar{\lambda}))} = \frac{\left\{\left[-\delta r^*\tau e^{-r^*\tau} + \delta\left(1 - e^{-r^*\tau}\right)\right]/r^{*2}\right\} + \tau e^{-r^*\tau}}{\left\{\left[-\delta r^* T e^{-r^*T} + \delta\left(1 - e^{-r^*T}\right)\right]/r^{*2}\right\} + T e^{-rT}} \frac{\psi_{ss}(T)}{\psi_{ss}(\tau)}$$
$$= \frac{1 + [-1 + (r^*/\delta - 1)r^*\tau]e^{-r^*\tau}}{1 + [-1 + (r^*/\delta - 1)r^* T]e^{-r^*T}} \frac{\psi_{ss}(T)}{\psi_{ss}(\tau)}.$$

If we define

$$\Xi \equiv \lim_{\bar{\lambda}\to 0} \iota_\infty(T, r_\infty(\bar{\lambda})),$$

then

$$\lim_{\bar{\lambda}\to 0} \iota_\infty(\tau, r_\infty(\bar{\lambda})) = \frac{1 + [-1 + (r^*/\delta - 1)r^*\tau]e^{-r^*\tau}}{1 + [-1 + (r^*/\delta - 1)r^* T]e^{-r^*T}} \frac{\psi_{ss}(T)}{\psi_{ss}(\tau)} \Xi.$$

The value of Ξ then must be consistent with zero consumption:

$$y_{ss} - f_\infty^{\bar{\lambda}\to 0}(0) + \int_0^T \left[\iota_\infty^{\bar{\lambda}\to 0}(\tau)\psi_{ss}(\tau) - \delta f_\infty^{\bar{\lambda}\to 0}(\tau)\right] d\tau = 0,$$

for $f_\infty^{\bar{\lambda}\to 0}(\tau) = \int_\tau^T \iota_\infty^{\bar{\lambda}\to 0}(s) ds$. QED

A7. Proof of Proposition 3

When $\delta = 0$, we have that

$$\iota_{ss}(\tau) = \frac{1}{\bar{\lambda}}\left(1 - \exp\left(-\left(\rho - r_{ss}^*\right)\tau\right)\right).$$

Define the spread, $\Delta \equiv \rho - r_{ss}^*$. We have that

$$\frac{\partial \iota_{ss}(\tau)}{\partial \Delta} = \frac{\partial}{\partial \Delta}\left[\frac{1}{\bar{\lambda}}(1 - \exp(-\Delta\tau))\right] = \frac{1}{\bar{\lambda}}(\tau \exp(-\Delta\tau)) > 0.$$

From here, we compute the elasticity with respect to relative impatience:

$$\epsilon_{ss,\Delta}^{\tau} = \frac{\Delta}{\iota_{ss}(\tau)} \frac{\partial \iota_{ss}(\tau)}{\partial \Delta} = \Delta \frac{\tau \exp(-\Delta\tau)}{1 - \exp(-\Delta\tau)} = \frac{\Delta\tau}{\exp(\Delta\tau) - 1} > 0.$$

[28] We drop the subindex "ss "to ease the notation.

Next, to employ lemma 1, we must determine whether this elasticity is increasing in τ. Note that

$$\frac{\partial}{\partial \tau}[\epsilon^{\tau}_{ss,\Delta}] = \frac{\Delta\tau}{\exp(\Delta\tau) - 1}\left(\frac{1}{\Delta\tau} - \frac{\exp(\Delta\tau)}{\exp(\Delta\tau) - 1}\right).$$

Thus, the elasticity is decreasing in τ if the term in parentheses is negative, that is, if

$$1 < \frac{\Delta\tau \exp(\Delta\tau)}{\exp(\Delta\tau) - 1}.$$

Rearranging,

$$\exp(\Delta\tau) < 1 + \Delta\tau \exp(\Delta\tau). \tag{A29}$$

To show that the inequality indeed holds, consider the function

$$z(x; \Delta\tau) = \exp(x \cdot \Delta\tau).$$

Since $z(x; \Delta\tau)$ is a (strictly) convex function for any $\Delta\tau > 0$, we have that

$$z(y; \Delta\tau) > z(x; \Delta\tau) + z_x(x; \Delta\tau)|_{x=1}(y - x),$$

for any $\{x, y\}$. In particular, for $x = 0$ and $y = 1$, we have that

$$z(0; \Delta\tau) > z(1; \Delta\tau) + z_x(1; \Delta\tau)|_{x=1}(0 - 1)$$

or, equivalently,

$$1 > \exp(\Delta\tau) - \Delta\tau \exp(\Delta\tau).$$

Rearranging terms yields condition (A29). Thus, by lemma 1, the WAM decreases with impatience. QED

A8. *Proof of Proposition 4*

A8.1. Part 1: Smoothing

We investigate the effect on the WAM of a temporary drop in steady state income to the initial income y_0. Thus, we consider a decline in income starting from the steady state at time zero. We investigate the special limit case, as liquidity costs are very large, $\bar{\lambda} \to \infty$, and bonds are zero-coupon, $\delta = 0$, which renders a closed-form solution. Note that in this case

$$\lim_{\bar{\lambda} \to \infty} f_{ss}(\tau) = 0, \quad \lim_{\bar{\lambda} \to \infty} \iota_t(\tau) = 0,$$

and thus

$$\lim_{\bar{\lambda} \to \infty} c_t = y_t.$$

Recall that the path of income is given by

$$y_t = y_{ss} + (y_0 - y_{ss})\exp(-\alpha t) \quad \text{and}$$
$$\dot{y}_t = -\alpha(y_0 - y_{ss})\exp(-\alpha t).$$

Now, consider a small negative initial drop in income near the steady state, $\varepsilon = y_{ss} - y_0 \gtrsim 0$. Therefore, we have (in the limit as $\varepsilon \to 0$)

$$\begin{aligned}\frac{\partial}{\partial \varepsilon}\left[\frac{\dot{y}_t}{y_t}\right]\Big|_{\epsilon=0} &= \frac{\partial}{\partial \varepsilon}\left[\frac{\alpha\varepsilon \exp(-\alpha t)}{y_{ss} - \varepsilon \exp(-\alpha t)}\right]\Bigg|_{\varepsilon=0} \\ &= \alpha \frac{\exp(-\alpha t)}{y_{ss} - \varepsilon \exp(-\alpha t)}\Bigg|_{\epsilon=0} + \frac{\alpha\varepsilon(\exp(-\alpha t))^2}{(y_{ss} - \varepsilon \exp(-\alpha t))^2}\Bigg|_{\varepsilon=0} \\ &= \alpha \frac{\exp(-\alpha t)}{y_{ss}}.\end{aligned}$$

Because we are working with the large-$\bar{\lambda}$ limit, we have

$$\lim_{\bar{\lambda}\to\infty} \frac{\partial}{\partial \varepsilon}\left[\frac{\dot{c}_t}{c_t}\right]\Bigg|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}\left[\frac{\dot{y}_t}{y_t}\right]\Bigg|_{\varepsilon=0} = \alpha \frac{\exp(-\alpha t)}{y_{ss}}.$$

From here, we can compute the impact on the domestic discount. Recall that

$$r_t = \rho + \sigma \cdot \frac{\dot{c}_t}{c_t}.$$

Thus, we have that

$$\lim_{\bar{\lambda}\to\infty} \frac{\partial}{\partial \varepsilon}[r_t] = \sigma\alpha\left[\frac{\exp(-\alpha t)}{y_{ss}}\right].$$

Now, recall that the optimal issuances at time zero are given by

$$\iota_0(\tau) = \frac{1}{\bar{\lambda}}\left(1 - \frac{v_0(\tau)}{\psi_0(\tau)}\right) = \frac{1}{\bar{\lambda}}\left(1 - \exp\left(-\int_0^\tau \left(r_s - r_{ss}^*\right) ds\right)\right) > 0.$$

Thus, we have that

$$\begin{aligned}\epsilon_{0,\varepsilon}^{\tau} &\equiv \frac{\partial \iota_0(\tau)}{\partial \varepsilon} \cdot \frac{1}{\iota_0(\tau)} \\ &= \frac{\partial\left(1 - \exp\left(-\int_0^\tau \left(r_s - r_{ss}^*\right) ds\right)\right)}{\partial \varepsilon} \cdot \frac{1}{\left(1 - \exp\left(-\int_0^\tau \left(r_s - r_{ss}^*\right) ds\right)\right)} \\ &= \left(-\frac{\exp\left(-\int_0^\tau r_s - r_{ss}^*\, ds\right)}{1 - \exp\left(-\int_0^\tau r_s - r_{ss}^*\, ds\right)}\right) \cdot \left(-\int_0^\tau \frac{\partial}{\partial \varepsilon}[r_s] ds\right) \\ &= \frac{\int (\partial/\partial\varepsilon)[r_s] ds}{\exp\left(\int_0^\tau r_s - r_{ss}^*\, ds\right) - 1} \frac{1}{y_{ss}}.\end{aligned}$$

Therefore, in this expression we have that

$$\lim_{\bar{\lambda}\to\infty}\epsilon_{0,\varepsilon}^{\tau} = \frac{1}{y_{ss}}\sigma\alpha\frac{\int_0^{\tau}\exp(-\alpha s)ds}{\exp((\rho - r^*)\tau) - 1} = -\frac{1}{y_{ss}}\sigma\frac{\exp(-\alpha\cdot s)|_{s=0}^{\tau}}{\exp((\rho - r^*)\tau) - 1}$$
$$= -\frac{1}{y_{ss}}\sigma\frac{\exp(-\alpha\tau) - 1}{\exp((\rho - r^*)\tau) - 1}.$$

Finally, note then that, reversing signs, we obtain

$$\lim_{\bar{\lambda}\to\infty}\epsilon_{0,\varepsilon}^{\tau} = \frac{\sigma}{y_{ss}}\cdot\frac{1 - \exp(-\alpha\tau)}{\exp((\rho - r^*)\tau) - 1} \gtrsim 0.$$

Thus, we have that issuances increase with the drop in income and scale with the IES coefficient. Next, we show that the WAM is decreasing with the perturbation. We need to show that $\epsilon_{0,\varepsilon}^{\tau}$ is decreasing in τ. Note that

$$\frac{\partial}{\partial\tau}[\epsilon_{0,\varepsilon}^{\tau}] = \epsilon_{0,\varepsilon}^{\tau}\left[\alpha\frac{\exp(-\alpha\tau)}{1 - \exp(-\alpha\tau)} - (\rho - r^*)\frac{\exp((\rho - r^*)\tau)}{\exp((\rho - r^*)\tau) - 1}\right].$$

Thus,

$$\operatorname{sign}\left(\frac{\partial}{\partial\tau}[\epsilon_{0,\varepsilon}^{\tau}]\right) = \operatorname{sign}\left(\alpha\frac{1}{\exp(\alpha\tau) - 1} - (\rho - r^*)\frac{1}{1 - \exp(-(\rho - r^*)\tau)}\right)$$
$$= \operatorname{sign}\left(\frac{\alpha}{\rho - r^*} - \frac{\exp(\alpha\tau) - 1}{1 - \exp(-(\rho - r^*)\tau)}\right).$$

Define

$$h(\tau) \equiv \frac{\exp(\alpha\tau) - 1}{1 - \exp(-(\rho - r^*)\tau)} > 0.$$

By L'Hôpital's rule, the function

$$\lim_{\tau\to 0}\frac{\exp(\alpha\tau) - 1}{1 - \exp(-(\rho - r^*)\tau)} = \frac{\alpha}{\rho - r^*}.$$

It suffices to show that $h(\tau)$ is increasing for $\tau > 0$ to show that the elasticities are decreasing. We show this by contradiction. Suppose that $h(\tau)$ is decreasing or constant, and hence that $h(\tau) \leq \alpha/(\rho - r^*)$. The derivative of this function is nonpositive, $h'(\tau) \leq 0$. Then,

$$h'(\tau) = \left[\frac{\alpha\exp(\alpha\tau)}{\exp(\alpha\tau) - 1} - \frac{(\rho - r^*)\exp(-(\rho - r^*)\tau)}{1 - \exp(-(\rho - r^*)\tau)}\right]\frac{\exp(\alpha\tau) - 1}{1 - \exp(-(\rho - r^*)\tau)}$$
$$= h(\tau)\left[\alpha\frac{1}{1 - \exp(-\alpha\tau)} - (\rho - r^*)\frac{1}{\exp((\rho - r^*)\tau) - 1}\right] \leq 0,$$

or

$$\frac{\alpha}{(\rho - r^*)} \leq \frac{1 - \exp(-\alpha\tau)}{\exp((\rho - r^*)\tau) - 1}.$$

This requires that

$$\frac{\exp(\alpha\tau)-1}{1-\exp(-(\rho-r^*)\tau)} = h(\tau) \le \frac{\alpha}{\rho-r^*} \le \frac{1-\exp(-\alpha\tau)}{\exp((\rho-r^*)\tau)-1}$$
$$= \frac{\exp(-\alpha\tau)}{\exp((\rho-r^*)\tau)}\frac{\exp(\alpha\tau)-1}{1-\exp(-(\rho-r^*)\tau)}$$

and simplifying

$$\exp((\rho-r^*)\tau) \le \exp(-\alpha\tau),$$

which is false for any $\tau > 0$, since $\rho > r^*$. Thus, since $h(\tau)$ is increasing, the elasticities $\epsilon^{\tau}_{0,\varepsilon}$ are decreasing in τ. Hence, by lemma 1, the WAM is decreasing in the perturbation, that is,

$$\lim_{\bar{\lambda}\to\infty}\epsilon^{\mu}_{0,\varepsilon} < 0.$$

A8.2. Part 2: Yield-Curve Riding

Recall that the path of international rates is given by

$$r^*_t = r^*_{ss} + \left(r^*_0 - r^*_{ss}\right)\exp(-\alpha t).$$

Now, consider a small initial increase in rates. Thus, we have $r^*_0 = r^*_{ss} + \varepsilon$. With zero coupons,

$$\psi_0(\tau) = \exp\left(-\int_0^{\tau} r^*_s\, ds\right) = \exp\left(-r^*_{ss}\tau - \varepsilon\cdot\int_0^{\tau}\exp(-\alpha t)ds\right).$$

We compute the integral to obtain

$$\begin{aligned}\psi_0(\tau) &= \exp\left(-r^*_{ss}\tau + \frac{\varepsilon}{\alpha}\cdot\int_0^{\tau}(-\alpha)\cdot\exp(-\alpha\cdot t)ds\right)\\ &= \exp\left(-r^*_{ss}\tau + \frac{\varepsilon}{\alpha}(\exp(-\alpha\cdot\tau)-1)\right)\\ &= \exp\left(-r^*_{ss}\tau\right)\exp\left(\frac{\varepsilon}{\alpha}(\exp(-\alpha\cdot\tau)-1)\right)\\ &= \psi_{ss}(\tau)\cdot\exp\left(\frac{\varepsilon}{\alpha}(\exp(-\alpha\cdot\tau)-1)\right).\end{aligned}$$

Next, the derivative (in the limit as $\varepsilon\to 0$) is given by

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}[\psi_0(\tau)]|_{\varepsilon=0} &= \psi_{ss}(\tau)\frac{1}{\alpha}(\exp(-\alpha\cdot\tau)-1)\exp\left(\frac{\varepsilon}{\alpha}(\exp(-\alpha\cdot\tau)-1)\right)\Big|_{\varepsilon=0}\\ &= \frac{1}{\alpha}\psi_{ss}(\tau)(\exp(-\alpha\cdot\tau)-1) < 0,\end{aligned}$$

and

$$\psi_0(\tau)|_{\varepsilon=0} = \psi_{ss}(\tau).$$

Now, recall that issuances are

$$\iota_0(\tau) = \frac{1}{\bar{\lambda}}\left(1 - \frac{v_0(\tau)}{\psi_0(\tau)}\right) > 0.$$

Thus, since $\sigma = 0$ implies $v_0(\tau) = \exp(-\rho\tau)$, we have that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}[\iota_0(\tau)]|_{\epsilon=0} &= -\frac{1}{\bar{\lambda}}\left(-\frac{v_0(\tau)}{\psi_0(\tau)}\frac{(\partial/\partial\varepsilon)[\psi_0(\tau)]}{\psi_0(\tau)}\right)\Bigg|_{\epsilon=0} \\ &= \frac{1}{\alpha\bar{\lambda}}\exp\left(-\left(\rho - r_{ss}^*\right)\tau\right)(\exp(-\alpha\cdot\tau) - 1)).\end{aligned}$$

Therefore,

$$\begin{aligned}\epsilon^{\tau}_{0,\varepsilon} &= \frac{1}{\iota_0(\tau)}\frac{\partial}{\partial\epsilon}[\iota_0(\tau)] = \frac{1}{\alpha}\frac{\exp\left(-\left(\rho - r_{ss}^*\right)\tau\right)(\exp(-\alpha\cdot\tau) - 1))}{1 - \left(\exp\left(-\left(\rho - r_{ss}^*\right)\tau\right)\right)} \\ &= -\frac{1}{\alpha}\frac{1 - \exp(-\alpha\cdot\tau)}{\exp\left(\left(\rho - r_{ss}^*\right)\tau\right) - 1}.\end{aligned}$$

In the proof of the effect of smoothing, we showed that the function $h(\tau)$ is increasing. Since $(1 - \exp(-\alpha\cdot\tau))/(\exp((\rho - r_{ss}^*)\tau) - 1) = -h(\tau)$, this term is decreasing. Multiplying it by the constant $(-1/\alpha)$ implies that $\epsilon^{\tau}_{0,\varepsilon}$ is increasing in τ. Thus, the WAM increases with a temporary positive increase in the level of interest rates—and the yield curve slopes downward. QED

A9. Duality

Given a path of resources y_t, the primal problem is given by

$$\begin{aligned}V[f_0(\cdot)] &= \max_{\{\iota_t(\tau),c_t\}_{t\in[0,\infty),\tau\in[0,T]}}\int_t^{\infty} e^{-\rho(s-t)}u(c(s))ds \quad \text{subject to} \\ c_t &= y_t - f_t(0) + \int_0^T [q_t(\tau,\iota)\iota_t(\tau) - \delta f_t(\tau)]d\tau \quad \text{and} \\ \frac{\partial f}{\partial t} &= \iota_t(\tau) + \frac{\partial f}{\partial\tau}.\end{aligned}$$

Here we show that this problem has a dual formulation. This dual formulation minimizes the resources needed to sustain a given path of consumption c_t:

$$\begin{aligned}D[f_0(\cdot)] &= \min_{\{\iota_t(\tau)\}_{t\in[0,\infty),\tau\in[0,T]}}\int_0^{\infty} e^{-\int_0^t r(s)\,ds} y_t\,dt \quad \text{subject to} \\ c_t &= y_t - f_t(0) + \int_0^T [q(\tau,t,\iota)\iota_t(\tau) - \delta f_t(\tau)]d\tau, \\ \frac{\partial f}{\partial t} &= \iota_t(\tau) + \frac{\partial f}{\partial\tau}, \quad \text{and} \\ r_t &= \rho - \frac{U''(c_t)c_t}{U'(c_t)}\frac{\dot{c}_t}{c_t}.\end{aligned}$$

PROPOSITION 9. Consider the solution $\{c_t^*, \iota_t^*(\tau), f_t^*(\tau)\}_{t\geq 0,\tau\in(0,T]}$ to the primal problem, given f_0. Then, given the path of consumption c_t^*, $\{y_t^*, \iota_t^*(\tau), f_t^*(\tau)\}_{t\in[0,\infty),\tau\in(0,T]}$ solves the dual problem where

$$y_t^* = c_t^* + f_t^*(0) + \int_0^T \left[q_t(\tau, \iota^*)\iota_t^*(\tau) - \delta f_t^*(\tau)\right] d\tau.$$

Proof. We start by following the steps of proposition 1. We construct the Lagrangian for the dual problem in the space $\| e^{-\rho t/2} g_t(\tau) \|^2 < \infty$. After replacement of the resources y_t needed to support a path of consumption c_t, the budget constraint is

$$\begin{aligned}\mathcal{L}[\iota, f] = &\int_0^\infty e^{-\int_0^t r_t\, ds}\left(c_t + f_t(0) - \int_0^T [q_t(\tau, \iota)\iota_t(\tau) - \delta f_t(\tau)]\, d\tau\right) dt \\ &+ \int_0^\infty \int_0^T e^{-\int_0^t r_t\, ds} v_t(\tau)\left(-\frac{\partial f}{\partial t} + \iota_t(\tau) + \frac{\partial f}{\partial \tau}\right) d\tau\, dt,\end{aligned}$$

where $v_t(\tau)$ is the Lagrange multiplier associated to the law of motion of debt. We again consider a perturbation $h_t(\tau)$, $e^{-\rho t} h \in L^2([0, T] \times [0, \infty))$, around the optimal solution. Recall that because f_0 is given and $f_t(T) = 0$, any feasible perturbation has to meet $h_0(\tau) = 0$ and $h_t(T) = 0$. At an optimal solution f, it must be the case that $\mathcal{L}[\iota, f] \geq \mathcal{L}[\iota, f + \alpha h]$ for any feasible perturbation $h_t(\tau)$. This implies that

$$\begin{aligned}\frac{\partial}{\partial \alpha}\mathcal{L}[\iota, f + \alpha h]|_{\alpha=0} = &\int_0^\infty e^{-\int_0^t r_t\, ds}\left[h_t(0) + \int_0^T \delta h_t(\tau)\, d\tau\right] dt \\ &- \int_0^\infty \int_0^T e^{-\int_0^t r_t\, ds} \frac{\partial h}{\partial t} v_t(\tau) d\tau\, dt \\ &+ \int_0^\infty \int_0^T e^{-\int_0^t r_t\, ds} \frac{\partial h}{\partial \tau} v_t(\tau) d\tau\, dt.\end{aligned}$$

We again employ integration by parts to show that

$$\begin{aligned}\int_0^\infty \int_0^T e^{-\int_0^t r_t\, ds} \frac{\partial h}{\partial t} v_t(\tau)\, d\tau\, dt &= \int_0^T \int_0^\infty e^{-\int_0^t r_t\, ds} v_t(\tau) \frac{\partial h}{\partial t} dt\, d\tau \\ &= \int_0^T (\lim_{s\to\infty} e^{-\int_0^t r_t\, ds} h_s(\tau) v_s(\tau)] - h_0(\tau) v_0(\tau)) d\tau \\ &\quad - \int_0^T \int_0^\infty e^{-\int_0^t r_t\, ds}\left(\frac{\partial v_t(\tau)}{\partial t} - r_t v_t(\tau)\right) h_t(\tau) dt\, d\tau \\ &= \int_0^T \left(\lim_{s\to\infty} e^{-\int_0^t r_t\, ds} h_s(\tau) v_s(\tau) - h_0(\tau) v_0(\tau)\right) d\tau \\ &\quad - \int_0^\infty e^{-\int_0^s r(u)\, du} \int_0^T \left(\frac{\partial v_t(\tau)}{\partial t} - r_t v_t(\tau)\right) h_t(\tau) d\tau\, dt,\end{aligned}$$

and

$$\int_0^\infty e^{-\int_0^t r_t\,ds}\int_0^T \frac{\partial h}{\partial \tau} v_t(\tau)d\tau\,dt = \int_0^\infty e^{-\int_0^t r_t\,ds}\left[h_t(T)v(T,t) - h_t(0)v_t(0) - \int_0^T h_t(\tau)\frac{\partial v}{\partial \tau}\,d\tau\right]dt.$$

Replacing these calculations in the Lagrangian and equating it to zero yields

$$\begin{aligned} 0 = & \int_0^\infty e^{-\int_0^t r_t\,ds}\left[h_t(0) + \int_0^T \delta h_t(\tau)\,d\tau\right]dt \\ & + \int_0^\infty\int_0^T e^{-\int_0^t r_t\,ds}\left(-r_t v - \frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial t}\right)h_t(\tau)d\tau\,dt \\ & + \int_0^\infty e^{-\int_0^t r_t\,ds}(h_t(T)v(T,t) - h_t(0)v_t(0))dt \\ & - \int_0^\infty \lim_{s\to\infty} e^{-\int_0^s r(u)\,du} h_s(\tau)v_s(\tau)d\tau. \end{aligned}$$

Again, the previous equation has to hold for any feasible variation $h_t(\tau)$, and all the terms that multiply $h_t(\tau)$ should be equal to zero. The latter yields a system of necessary conditions for the Lagrange multipliers, and substituting for the value of r_t:

$$\begin{aligned} \left(\rho - \frac{U''(c_t)c_t}{U'(c_t)}\frac{\dot{c}_t}{c_t}\right)v_t(\tau) &= \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \quad \text{if } \tau \in (0,T], \\ v_t(0) &= 1, \quad \text{if } \tau = 0, \\ \lim_{t\to\infty} e^{-\rho t}v_t(\tau) &= 0, \quad \text{if } \tau \in (0,T]. \end{aligned} \tag{A30}$$

By proceeding in a similar fashion with the control, we arrive at

$$\left(\frac{\partial q}{\partial \iota}\iota_t(\tau) + q_t(\tau,\iota)\right) = -v_t(\tau). \tag{A31}$$

Note that the system of equations (A30) and (A31), the budget constraint, the law of motion of debt, and initial debt f_0 together are precisely the conditions that characterize the solution of the primal problem. QED

Appendix B

Public Finance Considerations: A Public Finance Microfoundation

The goal of this appendix is to recast our original problem as a problem with distorting taxation. We modify the original model and let the government maximize the utility of households, who now also supply labor. Labor taxes are the only distorting tax. Government expenditures follow a deterministic path. The household utility is now

$$U\left(c_t - \chi \frac{h_t^{1+\nu}}{1+\nu}\right), \text{ where } U(x) \equiv \frac{x^{1-\sigma} - 1}{1-\sigma}.$$

In this case, h_t stands for hours worked and c_t for household consumption. For simplicity, we assume that output is linear in hours, setting the real wage to 1. The preferences are thus GHH, with χ a disutility scale parameter and ν the inverse Frisch elasticity. Households satisfy the following budget constraint:

$$c_t = (1 - \eta_t) \cdot h_t,$$

where η_t is a labor tax. We assume that the government saves on behalf of households. Also, the only possible way that the government can transfer resources to households is through tax subsidy. The problem of the household is static and thus is given by

$$\max_h U\left(c - \chi \frac{h^{1+\nu}}{1+\nu}\right), \quad \text{subject to } c = (1 - \eta_t) \cdot h.$$

Lifetime utility is given by

$$\int_0^\infty e^{-\rho t} U\left(c_t - \chi \frac{h_t^{1+\nu}}{1+\nu}\right) dt.$$

The tax receipts for the government are now given by

$$w_t = \eta_t \cdot h_t.$$

We assume that the government faces a known path of expenditures, g_t. We can assume that g_t is negative if the government has access to some endowment, for example, of natural resources. The government's budget constraint is

$$w_t + \int_0^T q_t(\iota, \tau)\iota(\tau, t) d\tau = g_t + \left[f_t(0) + \delta \int_0^T f_t(\tau) d\tau\right].$$

For convenience, note that if we define

$$y_t = -g_t$$

and set η_t to zero, we are back in the budget constraint in the main body of the paper. The difference from the original problem is that now we allow the government to potentially bear a negative value for y_t. This is possible because households can provide labor to make up for a negative value of y_t, something that is not possible in the original problem.

Definition 1. The problem of optimal maturity with distortionary labor taxes is

$$\max_{\{\eta_t, \iota_t(\tau)\}} \int_0^\infty e^{-\rho t} U\left(c_t - \chi \frac{h_t^{1+\nu}}{1+\nu}\right) dt,$$

subject to (i) that h is chosen optimally by the household, given η, (ii) the government budget constraint, and (iii) the law of motion of debt (eq. [8]), with an initial condition f_0.

Below, we prove that the solution to the problem with distortionary taxes is given by the solution to a modified version of the original problem without distorting taxes, where U is replaced by a modified return function. In particular, we prove the following result.

PROPOSITION 10. The solution to the problem of optimal maturity with distortionary labor taxes coincides with the solution to

$$\max_{\{\iota_t(\tau)\}}\int_0^\infty e^{-\rho t}U(W(x_t))dt,$$

subject to the budget constraint

$$x_t = -y_t + \left[f_t(0) + \delta\int_0^T f_t(\tau)\,d\tau\right] - \int_0^T q_t(\iota,\tau)\iota(\tau,t)d\tau$$

and the law of motion of debt (eq. [8]) with an initial condition f_0. The function W is given by

$$W(x) \equiv \left\{c|c - \chi^{-1/(1+\nu)}\cdot c^{1/(1+\nu)} = x\right\},$$

with domain in all x such that $W(x) \geq [(\chi^{-1/(1+\nu)}/(1+\nu)]^{(1+\nu)/\nu}$. Finally, given the path of x_t consistent with the solution to $\iota_t(\tau)$, the optimal consumption and labor, and taxes in 1 are given by

$$c_t = W(x_t),$$
$$h_t = \left(\frac{W(x_t)}{\chi}\right)^{1/(1+\nu)},$$

and

$$\eta_t = 1 - \chi\left(\frac{W(x_t)}{\chi}\right)^{\nu/(1+\nu)}.$$

Proposition 10 shows that problem 1 can be solved by first solving the problem without distortionary taxes with a modified objective function and then backing out the optimal taxes from the optimal-issuance rule. The following immediate corollary presents the solution presented in the body of the paper.

COROLLARY 1. The optimal-issuance rules in problem 1 are given by

$$\iota_t(\tau) = \frac{1}{\overline{\lambda}}\cdot\frac{\psi_t(\tau) - v_t(\tau)}{\psi_t(\tau)},$$

with

$$r_t v_t(\tau) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau},\ \tau\in(0,T],\ \text{and } v_t(0) = 1,$$

where

$$r_t = \rho + \left(\underbrace{\sigma}_{\text{intertemporal}} - \underbrace{\frac{W_t''}{W_t'}x_t}_{\text{intratemporal}}\right)\frac{\dot{x}_t}{x_t}$$

and

$$x_t = y_t + \left[f_t(0) + \delta\int_0^T f_t(\tau)d\tau\right] - \int_0^T q_t(\iota,\tau)\iota(\tau,t)d\tau.$$

B1. Proof of Proposition 10

The first step is to add the household's and government's budget constraints to obtain an aggregate budget constraint,

$$g_t + c_t + \left[f_t(0) + \delta\int_0^T f_t(\tau)d\tau\right] = h_t + \int_0^T q_t(\iota,\tau)\iota(\tau,t)d\tau.$$

Next, using that $g_t = -y_t$, we have that

$$x_t \equiv y_t + \int_0^T q_t(\iota,\tau)\iota(\tau,t)d\tau - \left[f_t(0) + \delta\int_0^T f_t(\tau)d\tau\right],$$

where x_t stands for consumption minus output. That is, this variable stands for change in national wealth, in other words, the current-account deficit.

The second step is to solve the household's problem. The household first-order conditions are

$$(1 - \eta_t) = h_t^{\nu},$$

and recall that the household's budget constraint is

$$c_t = (1 - \eta_t)h_t.$$

Combining the two equations above, we obtain the following relations:

$$c_t = (1 - \eta_t)h_t = h_t^{\nu+1}.$$

Thus, in equilibrium, the term inside the household's utility can be expressed solely in terms of consumption. To see this, note that

$$c_t - \frac{h_t^{1+\nu}}{1+\nu} = c_t - \frac{c_t}{1+\nu} = \frac{\nu}{1+\nu}\cdot c_t.$$

Therefore, we have that the immediate household utility is given by

$$U\left(c_t - \chi\frac{h_t^{1+\nu}}{1+\nu}\right) = \left(\frac{\nu}{1+\nu}\right)^{1-\sigma}U(c_t).$$

We can ignore the scale when we move back to solving the objective function.

The third step is to map x_t to a value for c_t. For that, we observe that

$$c^{1/(1+\nu)} = h;$$

thus, we have that

$$x = \Gamma(c) \equiv c - c^{1/(1+\nu)}.$$

The interpretation of $\Gamma(c)$ is that it maps a level of consumption, together with an equilibrium labor market choice, to a level of the current-account deficit. QED

B2. Shape of Γ

Next, we investigate the shape of $\Gamma(c)$. This function satisfies

$$\Gamma'(c) = 1 - \frac{c^{-\nu/(1+\nu)}}{1+\nu} \text{ and } \Gamma''(c) = \nu \frac{c^{(-1-2\nu)/(1+\nu)}}{(1+\nu)^2} > 0.$$

Thus, it is a convex function. Therefore, it has a unique minimum, which is achieved at

$$\bar{c} = \left(\frac{1}{1+\nu}\right)^{(1+\nu)/\nu}$$

and has roots at $c = \{0, 1\}$. Hence, the function is increasing in x_t in the region $c > \bar{c}$. Then, we can obtain the maximum value of Γ,

$$\Gamma(\bar{c}) = \left(\frac{1}{1+\nu}\right)^{(1+\nu)/\nu} - \left(\left(\frac{1}{1+\nu}\right)^{(1+\nu)/\nu}\right)^{1/(1+\nu)},$$

and thus obtain

$$\bar{x} \equiv \Gamma(\bar{c}) = \left(\frac{1}{1+\nu}\right)^{(1+\nu)/\nu} - \left(\frac{1}{1+\nu}\right)^{1/\nu} = -\left(\frac{1}{1+\nu}\right)^{1/\nu}\left(\frac{\nu}{1+\nu}\right) \leq 0.$$

Past $\bar{c}$, the function is decreasing. Thus, for any $c \geq \bar{c}$ we can define the inverse:

$$W(x) = \Gamma^{-1}(x) \quad \text{for } x \geq \bar{x}.$$

The inverse is increasing in the region. Now, observe that for any $x \geq \bar{x}$ we can map x to a value of consumption.

Next, observe that if the government ever reaches a point where $x < \bar{x}$, the government cannot induce a higher current-account surplus. To induce a higher current-account surplus, households need to work more, but to do so, they need a greater wage subsidy. The issue is that past that subsidy, the leisure disutility income effect is so large that it induces more consumption. Thus, $\bar{x}$ is as a satiation point for the government. Thus, an optimum solution to the government problem will restrict the solution such that $x_t \geq \bar{x}$ at all t. Thus, $\bar{x}$ is the peak of a Laffer curve in this model.

Finally, observe that we have

$$U\left(c_t - \chi\frac{h_t^{1+\nu}}{1+\nu}\right) = \left(\frac{\nu}{1+\nu}\right)^{1-\sigma} U(c_t) = \left(\frac{\nu}{1+\nu}\right)^{1-\sigma} U(W(x_t))$$

when the labor market is at equilibrium, for $x_t \geq \bar{x}$. Thus, the objective of the government in the modified problem is

$$V(f_0) = \max_{\{\iota_t(\tau)\}} \left(\frac{\nu}{1+\nu}\right)^{1-\sigma} \int_0^\infty e^{-\rho t} U(U(W(x_t)))dt,$$

where

$$x_t = y_t + \int_0^T q_t(\iota, \tau)\iota(\tau, t)d\tau - \left[f_t(0) + \delta\int_0^T f_t(\tau)d\tau\right],$$

with the restriction that $x_t \geq \bar{x}$. The problem is identical to the original version of the problem without labor taxes. Thus, their solutions must coincide.

B3. Optimal Issuances

Define now

$$Z(x) = U(W(x)).$$

Thus, Z is the indirect utility associated with a current-account deficit. Next, note that

$$r_t = \rho - \frac{(Z'')}{Z'} x \cdot \frac{\dot{x}}{x},$$

as we showed in the body of the text. Then, we have that

$$Z' = U'W' \quad \text{and} \quad Z'' = U''W' + U'W''.$$

Therefore,

$$r_t = \rho - \frac{(U'' \cdot W' + U'W'')}{U'W'} x \cdot \frac{\dot{x}}{x} = \rho - \frac{U''}{U'} W \frac{x}{W} \frac{\dot{x}}{x} - \frac{W''}{W'} W \frac{1}{W} \frac{\dot{x}}{x},$$

and note that

$$\sigma = -\frac{U''}{U'} W.$$

Thus, we have that

$$r_t = \rho + \left(\sigma \frac{x}{W} - \frac{W''}{W'}\right) \frac{\dot{x}}{x}.$$

The rest of the formulas are as before:

$$\iota_t(\tau) = \frac{1}{\bar{\bar{\lambda}}} \cdot \frac{\psi_t(\tau) - v_t(\tau)}{\psi_t(\tau)},$$

and

$$r_t v_t(\tau) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \ \tau \in (0, T], \text{ and } v_t(0) = 1.$$

To conclude, we present a map from x_t to the allocations:

$$c_t = W(x_t), \ h_t = (c_t)^{1/(1+\nu)} = W(x_t)^{1/(1+\nu)}, \text{ and } \eta_t = 1 - W(x_t).$$

Appendix C

Computational Method

We provide here a sketch of the numerical algorithm used to jointly solve for the equilibrium domestic valuation, $v_t(\tau)$, bond price, $q(t, \tau, \iota)$, consumption c_t, issuance $\iota_t(\tau)$, and density f_t in the perfect-foresight case. The initial distribution is $f_0(\tau)$. The algorithm proceeds in three steps. We describe each step in turn.

C1. Step 1: Solution to the Domestic Value

The steady-state equation (14) is solved using an "upwind finite difference" scheme similar to Achdou et al. (2022). We approximate the valuation $v_{ss}(\tau)$ on a finite grid with step $\Delta\tau : \tau \in \{\tau_1, ..., \tau_I\}$, where $\tau_i = \tau_{i-1} + \Delta\tau = \tau_1 + (i-1)\Delta\tau$ for $2 \leq i \leq I$. The bounds are $\tau_1 = \Delta\tau$ and $\tau_I = T$, such that $\Delta\tau = T/I$. We use the notation $v_i \coloneqq v_{ss}(\tau_i)$, and similarly for the issuance ι_i,. Note first that the domestic valuation equation involves first derivatives of the valuations. At each point of the grid, the first derivative can be approximated with a forward or a backward approximation. In an upwind scheme, the choice of forward or backward derivative depends on the sign of the drift function for the state variable. As in our case the drift is always negative, we employ a backward approximation in state

$$\frac{\partial v(\tau_i)}{\partial \tau} \approx \frac{v_i - v_{i-1}}{\Delta\tau}. \tag{C1}$$

The equation is approximated by the following upwind scheme,

$$\rho v_i = \delta + \frac{v_{i-1}}{\Delta\tau} - \frac{v_i}{\Delta\tau},$$

with terminal condition $v_0 = v(0) = 1$. This can be written in matrix notation as

$$\rho\mathbf{v} = \mathbf{u} + \mathbf{A}\mathrm{v},$$

where

$$\mathbf{A} = \frac{1}{\Delta\tau}\begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{I-1} \\ v_I \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \delta - 1/\Delta\tau \\ \delta \\ \delta \\ \vdots \\ \delta \\ \delta \end{bmatrix}. \tag{C2}$$

The solution is given by

$$\mathbf{v} = (\rho\mathbf{I} - \mathbf{A})^{-1}\mathbf{u}. \tag{C3}$$

Most computer software packages, such as Matlab, include efficient routines to handle sparse matrices such as **A**.

To analyze the transitional dynamics, define $t^{\max}$ as the time interval considered, which should be large enough to ensure a convergence to the stationary distribution, and time is discretized as $t_n = t_{n-1} + \Delta t$, in intervals of length

$$\Delta t = \frac{t^{\max}}{N - 1},$$

where N is a constant. We use now use the notation $v_i^n := v_{t_n}(\tau_i)$. The valuation at $t^{\max}$ is the stationary solution, computed in equation (C3), that we denote as $\mathbf{v}^N$. We choose a forward approximation in time. The dynamic value equation (14) can thus be expressed

$$r^n \mathbf{v}^n = \mathbf{u} + \mathbf{A}\mathbf{v}^n + \frac{(\mathbf{v}^{n+1} - \mathbf{v}^n)}{\Delta t},$$

where $r^n := r(t_n)$. By defining $\mathbf{B}^n = ((1/\Delta t) + r^n)\mathbf{I} - \mathbf{A}$ and $\mathbf{d}^{n+1} = \mathbf{u} + (\mathbf{v}^{n+1}/\Delta t)$, we have

$$\mathbf{v}^n = (\mathbf{B}^n)^{-1}\mathbf{d}^{n+1}, \tag{A4}$$

which can be solved backward from $n = N - 1$ until $n = 1$.

The optimal issuance is given by

$$\iota_i^n = \frac{1}{\lambda}\frac{(\psi_i^n - v_i^n)}{\psi_i^n},$$

where ψ_i^n is computed in a form analogous to v_i^n.

C2. Step 2: Solution to the Kolmogorov Forward Equation (KFE)

Analogously, the KFE of equation (8) can be approximated as

$$\frac{f_i^n - f_i^{n-1}}{\Delta t} = \iota_i^n + \frac{f_{i+1}^n - f_i^n}{\Delta \tau},$$

where we have employed the notation $f_i^n := f_{t_n}(\tau_i)$. This can be written in matrix notation as

$$\frac{\mathbf{f}^n - \mathbf{f}^{n-1}}{\Delta t} = \mathbf{i}^n + \mathbf{A}^{\mathrm{T}}\mathbf{f}^n, \tag{A5}$$

where $\mathbf{A}^{\mathrm{T}}$ is the transpose of **A** and

$$\mathbf{f}^n = \begin{bmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_{I-1}^n \\ f_I^n \end{bmatrix}, \quad \mathbf{i}^n = \begin{bmatrix} \iota_1^n \\ \iota_2^n \\ \vdots \\ \iota_{I-1}^n \\ \iota_I^n \end{bmatrix}.$$

Given $\mathbf{f}_0$, the discretized approximation to the initial distribution $f_0(\tau)$, we can solve the KFE as

$$\mathbf{f}_n = (\mathbf{I} - \Delta t \mathbf{A}^{\mathrm{T}})^{-1}(\mathbf{i}^n \Delta t + \mathbf{f}_{n-1}), \quad n = 1, \ldots, N. \tag{C6}$$

C3. Step 3: Computation of Expenditure

The discretized budget constraint (eq. [9]) can be expressed as

$$c^n = \bar{y}^n - f_1^{n-1} + \sum_{i=1}^{I}\left[\left(1_n - \frac{1}{2}\bar{\lambda}\iota_i^n\right)\iota_i^n \psi_i^n - \delta f_i^n\right]\Delta\tau, \quad n = 1, \ldots, N.$$

Compute

$$r^n = \rho + \frac{\sigma}{c^n}\frac{c^{n+1} - c^n}{\Delta t}, \quad n = 1, \ldots, N - 1.$$

C4. Complete Algorithm

The algorithm proceeds as follows. First, guess an initial path for consumption, for example $c^n = \bar{y}^n$, for $n = 1, .., N$. Set $k = 1$.

Step 1: Issuances.—Given c_{k-1}, solve step 1 and obtain ι.

Step 2: KFE.—Given ι, solve the KFE with initial distribution f_0 and obtain the distribution f.

Step 3: Consumption.—Given ι and f, compute consumption c. If $\| c - c_{k-1} \| = \sum_{n=1}^{N} |c^n - c_{k-1}^n| < \varepsilon$, then stop. Otherwise, compute

$$c_k = \omega c + (1 - \omega) c_{k-1}, \quad \lambda \in (0, 1),$$

set $k := k + 1$, and return to step 1.

Appendix D

Tables

TABLE D1
NONPARAMETRIC REGRESSION

Variables	Whole Sample (1)	Until 2010 (2)	From 2012 Onward (3)
$\alpha(3)$	−21.46**	−39.98**	−10.41*
	(10.63)	(20.24)	(6.12)
$\alpha(60)$	−14.82	15.46	−14.98
	(14.63)	(26.97)	(10.63)
$\alpha(120)$	−42.43***	12.05	−46.17***
	(13.89)	(24.13)	(11.83)
$\alpha(180)$	−74.25***	−38.09	−86.69***
	(18.41)	(43.40)	(17.05)
$\alpha(360)$	−70.33***	21.27	−123.70***
	(19.28)	(31.62)	(22.11)
$\lambda(36)$	8.01	11.06	5.73
	(5.28)	(8.55)	(3.65)
$\lambda(60)$	12.24***	10.46	12.71**
	(4.38)	(7.40)	(5.37)

TABLE D1 (*Continued*)

Variables	Whole Sample (1)	Until 2010 (2)	From 2012 Onward (3)
λ(120)	16.89***	10.62*	14.03***
	(3.55)	(6.31)	(4.22)
λ(180)	39.55***	28.27	44.77***
	(10.47)	(17.50)	(10.66)
λ(360)	55.63***	45.79***	59.72***
	(9.98)	(13.14)	(14.26)
Observations	1,143	337	627
R^2	.21	.23	.22

NOTE.—The dependent variable is the markup of the auction i on date t computed as $(\psi_{i,t} - q_{i,t})/\psi_{i,t}$, where $q_{i,t}$ is marginal price of the auction and $\psi_{i,t}$ is the closing secondary-market price on the same day. We drop observations with an issuance of less than 1 million euros. We include both competitive and noncompetitive auctions. All regressions include quarterly fixed effects. Column 1 reports the main specification corresponding to the full sample; col. 2 includes all issuances up to 2010 and col. 3 issuances after 2012. Robust standard errors are in parentheses.

* Statistically different from zero at the 10% confidence level.

** Statistically different from zero at the 5% confidence level.

*** Statistically different from zero at the 1% confidence level.

TABLE D2

PARAMETRIC REGRESSION

Variables	Whole Sample (1)	Until 2010 (2)	From 2012 Onward (3)
$\alpha(36)$	−20.75***	−32.86**	−12.52*
	(7.76)	(14.89)	(7.50)
$\alpha(60)$	−13.44	10.29	−27.18***
	(8.43)	(14.50)	(9.12)
$\alpha(120)$	−46.52***	−5.39	−66.28***
	(9.56)	(15.01)	(10.84)
$\alpha(180)$	−60.48***	−33.37	−78.06***
	(13.75)	(31.15)	(14.63)
$\alpha(360)$	−65.74***	21.23	−123.30***
	(17.82)	(28.93)	(21.40)
Issuance (% monthly GDP)	2.58	3.54	1.32
	(4.18)	(6.28)	(5.45)
Issuance × years to maturity	1.66***	1.20**	1.78***
	(.36)	(.47)	(.49)
Observations	1,143	337	734
R^2	.21	.22	.23

NOTE.—The dependent variable is the markup of the auction i on date t computed as $(\psi_{i,t} - q_{i,t})/\psi_{i,t}$, where $q_{i,t}$ is marginal price of the auction and $\psi_{i,t}$ is the closing secondary-market price on the same day. We drop observations with an issuance of less than 1 million euros. We include both competitive and noncompetitive auctions. All regressions include quarterly fixed effects. Column 1 reports the main specification corresponding to the full sample; col. 2 includes all issuances up to 2010 and col. 3 issuances after 2012. Robust standard errors are in parentheses.

* Statistically different from zero at the 10% confidence level.

** Statistically different from zero at the 5% confidence level.

*** Statistically different from zero at the 1% confidence level.

TABLE D3
ISSUANCE AND MATURITY DRIVERS

Variables	Issuances/GDP (quarterly %) (1)	Average Maturity (issuances) (2)
Constant	3.74**	109.90***
	(1.78)	(14.57)
Deficit/GDP (quarterly, %)	.65***	−.90
	(.12)	(.88)
Debt due/GDP (quarterly, %)	.70***	−.40
	(.09)	(.79)
Short-term rate factor (%)	−.34	−8.79***
	(.30)	(2.75)
Slope factor (%)	.40	−6.66**
	(.39)	(2.99)
Observations	80	66
R^2	.85	.25

NOTE.—The dependent variables are total issuances over quarterly GDP (col. 1) and the average WAM of issuances during the quarter (col. 2). The short-term rate and the slope factors are computed from the yields of Spanish debt and were provided to us by Jens Christensen. Robust standard errors are in parentheses.
** Statistically different from zero at the 5% confidence level.
*** Statistically different from zero at the 1% confidence level.

Appendix E

Equivalence between PDE and Integral Formulations

Valuations and prices are given by continuous-time net present value formulas. Their PDE representation is the analogue of the recursive representation in discrete time, and the integral formulation is the equivalent of the sequence summations. The solutions to each PDE can be recovered easily via the method of characteristics or as an immediate application of the Feynman-Kac formula. All of the PDEs in this paper have an exact solution contained in table E1.

TABLE E1
EQUIVALENCE BETWEEN PDE AND INTEGRAL FORMULATIONS

Price:	
PDE	$r^*{}_t\psi_t(\tau) = \delta + (\partial\psi/\partial t) - (\partial\psi/\partial\tau); \psi(0, t) = 1$
Integral	$e^{-\int_t^{t+\tau} r^*(u)\,du} + \delta\int_t^{t+\tau} e^{-\int_t^s r^*(u)\,du}\,ds$
Valuation:	
PDE	$r_t v_t(\tau) = \delta + (\partial v/\partial t) - (\partial v/\partial\tau); v(0, t) = 1$
Integral	$e^{-\int_t^{t+\tau} r(u)\,du} + \delta\int_t^{t+\tau} e^{-\int_t^s r(u)\,du}\,ds$
Debt profile:	
PDE	$\partial f/\partial t = \iota_t(\tau) + (\partial f/\partial\tau)$
Integral	$f_t(\tau) = \int_\tau^{\min\{T,\tau+t\}} \iota(s, t+\tau-s)\,ds + \mathbb{1}[T > t+\tau]\cdot f(0, \tau+t)$

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